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NSWC/DL TR-3726

COMPUTATION OF VELOCITY IN THE WAVE TRAIN OF A POINT SOURCE

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SEPTEMBER 1977

Approved for public release; distribution unlimited

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14 NSWC/DL-TR-3720

11 Sep[redacted]77

12 54p.

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Science and Mathematics Research Group

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FOREWORD

In this report there are presented efficient methods for the computation of velocity in the wave train of a point source. Part of the material in the report has been taken from previous reports and part has been developed in recent work. The report is intended to provide a systematic coherent documentation for subroutines. The manuscript was completed by 7 September 1977.

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ABSTRACT

The velocity potential and the components of velocity in the wave train of a point source are derived from Havelock's integral. Analysis and documentation are given for the computation of velocity by trapezoidal integration, by asymptotic approximation, and by integration by parts. Accuracy and efficiency are least dependent upon the depth of the source in the integration by parts. The three methods of computation are available for applications in a set of subroutines.

INTRODUCTION

In the computation of the flow around a ship, it is assumed usually that the fluid is incompressible and irrotational. It is assumed that the fluid is infinite in depth, and that the free surface can be represented by a linear approximation. In one method for the computation of the flow around a ship, the product of source density at the surface of the ship and the velocity of flow from a point source is integrated over the surface of the ship.

The classical approach to a surface disturbance by Kelvin[1] was an application of the principle of stationary phase to a determination of the first term of an asymptotic expansion Consideration of the possibility of development into full expansions has been made for the line of wave crests by Peters[2] and by Ursell[3]. Development of the full asymptotic approximation has been completed in the meantime[17]. The utilization of the asymptotic approximation is limited to the far field.

Fourier integrals for the velocity potential in the wave train of a point source have been derived for infinite depth by Havelock[4] and for finite depth by Lunde[5]. The components of velocity are known also as the derivatives of Green's function as defined by Wehausen[6].

The velocity in the flow from a point source is expressed as the sum of three terms. The first term is the flow from the source in an unbounded fluid, the second term is the flow from an image source above the free surface, and the third term is the flow in the wave train. The third term often is expressed as the sum of a single integral which represents a free wave, and a double integral which represents a bound wave. The single integral by itself satisfies the linearized free-boundary conditions, while the double integral combines with the first two terms to satisfy the linearized free-boundary conditions. The amount by which the single integral should be added can be determined by an analysis of the transient wave pattern of a source which starts at zero time and moves thereafter at a constant rate.

Components of velocity in physical space are derived from Fourier integration in wave number space. Either Cartesian coordinates or polar coordinates may be used in either space for the expression of functions. The usual choice is Cartesian coordinates in physical space and polar coordinates in wave number space.

The single integral is just the contribution from integration on a small circle at a singularity in the complex plane of the radial coordinate. The double integral is the Cauchy principal value of an integral whose integrand is infinite at the singularity. The single integral and the double integral can be combined into one complex integral in the complex plane of the radial coordinate. The integrand of the complex integral is analytic along the path of integration. It is necessary that the integrand be analytic, because the integral is the steady state limit of the transient wave which has an analytic integrand. It is not necessary to resort to ficticious frictions which have no physical meaning in an inviscid fluid.

Wherever the integrand is analytic, then in accordance with the Cauchy theorem, the path of integration may be deformed into any other path with the same limits of integration. The path of integration with respect to either coordinate may be moved out into the complex plane for that coordinate. Complex paths have been proposed several times in the literature[7-9]. Many years ago it was recognized that radial integration in wave number space is equivalent to the evaluation of the complex exponential integral for which we have standard methods of computation.

Even after the radial integration in wave number space, there remains an azimuthal integration in wave number space. The integrand is cyclical with respect to azimuth angle, and the trapezoidal rule is the high accuracy rule for numerical quadrature. The integrand is partly monotonic and partly oscillatory, and a large number of intervals is required by the trapezoidal rule, unless the integration is for a position in the neighborhood of the source.

A breakthrough was achieved when the quadrature was replaced with an integration by parts right through any number of fluctuations of the integrand. Convergent approximation is used in connection with the integration by parts. The convergent approximation is useful for both the near field and the far field.

The evaluation of the Green function and its derivatives has received enough attention in the literature to justify the release of formal subroutines for the computation of velocity potential and the components of velocity in the wave train of a point source. Such subroutines have been under development in this laboratory[15-17] for a number of years. The present report is a concise documentation of the assumptions and the formulations which are basic to the subroutines Important parts of the documentation are data on accuracy and efficiency.

The specification of accuracy for a subroutine depends upon whether the subroutine is intended for a specific application or for general utilization. The level of accuracy in the present project is eight decimal digits, while the computer essentially is a CDC 6600. In single precision this computer operates on 48-bit numbers. Most computers operate on numbers with a fixed number of binary bits. A gauge of accuracy for arithmetic operations is one unit in the lowest order bit. Larger errors accumulate during the computation of a function. There is inherent error which arises from error in the argument and there are truncation errors and rounding errors which arise from inaccuracy of computation. A sensible policy is to keep the truncation errors and the rounding errors to the same level as the inherent errors.

The absolute accuracy of a variable is expressed by the actual value of the error in the variable, while the relative accuracy of the variable is expressed by the ratio between the value of the error and the value of the variable. Whether the specification of accuracy can be a uniform absolute error or a uniform relative error depends upon the nature of the function. A uniform absolute error is appropriate for a periodic function, but a uniform relative error is appropriate for a monotonic function. For more complicated functions the absolute error may be a constant fraction of a reference function which matches the computed unction only at a limited number of maxima.

The number of cycles of a computation loop influences the accuracy of computation. The loop may be terminated when a sum becomes constant, when a preset tolerance is met, or when a preset number of cycles have occurred. Termination for a constant sum gives the full accuracy of the computer but may require an expensive sensing operation. More economy is possible if a simple empirical formula can be used to compute the number of cycles, but the derivation of an empirical formula may require a program of test runs on a computer.

In the computation of the components of velocity the gauge of accuracy is the square root of the sum of squares of the errors in the three components of velocity. At small orders or arguments the error diminishes smoothly with order or argument and is dominated by truncation error. At large orders or arguments the error fluctuates with an amplitude independent of order or argument and is dominated by rounding error. Transitions between modes of computation are designed so as to balance truncation error against rounding error.

Empirical curves have been constructed to express the error and the time for each of three methods of computation. The curves can indicate only trends in error or time because they are based upon actual runs on the CDC 6600 computer with time sharing and optimization. The error is subject to statistical fluctuations and the time depends upon the presence of other jobs in the computer. Under the conditions of the actual runs the cost of computation was 17¢ per second, but in other installations the cost would differ in accordance with costing algorithms.

CARTESIAN COORDINATES

Various authors on ship hydrodynamics have used various coordinate systems. The choice of coordinates for the present analysis is the standard coordinate system in aeronautics and oceanography. The forward direction is the direction of motion of the source through still fluid. The origin of coordinates is above the source at the undisturbed surface of the fluid. The x-coordinate is the distance forward, the y-coordinate is the distance to the right, and the z-coordinate is the distance downward. The arrangement of the Cartesian coordinates is illustrated in Figure 1.

FREE SURFACE

In accordance with the usual assumption of incompressible, irrotational flow, the velocity in the fluid is the negative gradient of a velocity potential which satisfies Laplace's equation. Under steady state conditions the potential in the moving reference frame is given by the expression

$$Ux + \varphi(x, y, z) \tag{1}$$

where Ux is the potential for uniform flow in a direction opposite to the motion of the source and $\varphi(x, y, z)$ is the potential for the local disturbance in the wave train. The Cartesian components u, v, w of local velocity are given by the equations

$$u = -\frac{\partial\varphi}{\partial x} \qquad v = -\frac{\partial\varphi}{\partial y} \qquad w = -\frac{\partial\varphi}{\partial z} \tag{2}$$

Let the configuration of the free surface be expressed by the equation

$$z + \zeta(x, y) = 0 \tag{3}$$

For steady-state conditions the velocity at the free surface is tangent to the surface and the potential obeys the boundary equation

$$\left(U + \frac{\partial\varphi}{\partial x}\right)\frac{\partial\zeta}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial\zeta}{\partial y} + \frac{\partial\varphi}{\partial z} = 0 \tag{4}$$

Neglect of terms of second order leads to the boundary equation

$$U\frac{\partial\zeta}{\partial x} + \frac{\partial\varphi}{\partial z} = 0 \tag{5}$$

The motion of the fluid is determined by the Bernoulli equation

$$\frac{1}{2}\left\{\left(U+\frac{\partial\varphi}{\partial x}\right)^2+\left(\frac{\partial\varphi}{\partial y}\right)^2+\left(\frac{\partial\varphi}{\partial z}\right)^2\right\}+\frac{p}{\rho}-gz=\tfrac{1}{2}U^2 \tag{6}$$

where ρ is the density, p is the pressure, and g is the acceleration of gravity. At the free surface the pressure is constant. Neglect of terms of second order leads to the boundary equation

$$U\frac{\partial\varphi}{\partial x}+g\zeta=0 \tag{7}$$

The free surface is eliminated from Equations (5) and (7) by differentiation to give the equation

$$\frac{\partial^2\varphi}{\partial x^2}-\kappa_0\frac{\partial\varphi}{\partial z}=0 \tag{8}$$

where κ_0 is a critical wave number and is defined by the equation

$$\kappa_0=\frac{g}{U^2} \tag{9}$$

Along the centerline behind the source the wave length λ of the transverse waves is given by the equation

$$\lambda=\frac{2\pi}{\kappa_0} \tag{10}$$

Inasmuch as the velocity field is symmetric with respect to the x-axis, it is sufficient to limit the analysis to positive values of y.

FOURIER ANALYSIS

The computation is founded on the assumption that velocity can be expressed by a Fourier transform. The two-dimensional Fourier transform is given by the equations

$$A(\alpha,\beta)=\frac{1}{4\pi^2}\iint F(x,y)\,e^{-i(\alpha x+\beta y)}\,dx\,dy \tag{11}$$

$$F(x,y)=\iint A(\alpha,\beta)\,e^{i(\alpha x+\beta y)}\,d\alpha\,d\beta \tag{12}$$

where x, y are Cartesian coordinates in physical space, α, β are Cartesian coordinates in wave number space, $F(x,y)$ is a function in physical space, and $A(\alpha,\beta)$ is the Fourier amplitude in wave number space. A potential $\varphi(x,y,z)$ of a source distribution on the plane $z=h$ can be constructed with the equation

$$\varphi(x,y,z)=\iint A(\alpha,\beta)\,e^{-\sqrt{\alpha^2+\beta^2}|z-h|+i(\alpha x+\beta y)}\,d\alpha\,d\beta \tag{13}$$

which gives a solution of Laplace's equation wherever $z\neq h$. The vertical component of velocity at $z = h$ is given by the equation

$$-\frac{\partial\varphi}{\partial z} = \pm \iint \sqrt{\alpha^2 + \beta^2}\, A(\alpha, \beta)\, e^{i(\alpha x + \beta y)}\, d\alpha\, d\beta \qquad (z \to h) \tag{14}$$

where the sign is plus if $z > h$ and the sign is minus if $z < h$. In accordance with the Gauss theorem the difference in $-\partial\varphi/\partial z$ on opposite sides of the plane $z = h$ is $4\pi\sigma(x, y)$ where $\sigma(x, y)$ is the surface source density at the plane. Thus the Fourier amplitude for an arbitrary distribution of source density is given by the equation

$$A(\alpha, \beta) = \frac{1}{2\pi\sqrt{\alpha^2 + \beta^2}} \iint \sigma(x, y)\, e^{-i(\alpha x + \beta y)}\, dx\, dy \tag{15}$$

For a unit source at the origin the Fourier amplitude is given just by the equation

$$A(\alpha, \beta) = \frac{1}{2\pi\sqrt{\alpha^2 + \beta^2}} \tag{16}$$

The amplitude varies inversely as the radial distance in wave number space.

Let κ, θ be polar coordinates in wave number space. The coordinates are related by the equations

$$\alpha = \kappa \cos\theta \qquad\qquad \beta = \kappa \sin\theta \tag{17}$$

The two-dimensional Fourier transform is given by the equations

$$A(\kappa, \theta) = \frac{1}{4\pi^2} \iint F(x, y)\, e^{-i\kappa(x\cos\theta + y\sin\theta)}\, dx\, dy \tag{18}$$

$$F(x, y) = \iint A(\kappa, \theta)\, e^{i\kappa(x\cos\theta + y\sin\theta)}\, \kappa\, d\kappa\, d\theta \tag{19}$$

For the unit source at the origin the Fourier amplitude is given by the equation

$$A(\kappa, \theta) = \frac{1}{2\pi\kappa} \tag{20}$$

The potential $\varphi_1(x, y, z)$ for the isolated unit source is given by the equation

$$\varphi_1(x, y, z) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \int_0^{\infty} e^{-\kappa|z-h| + i\kappa(x\cos\theta + y\sin\theta)}\, d\kappa\, d\theta \tag{21}$$

The potential of the point source does not satisfy the free-boundary conditions. An additional potential is added in order to meet the free-boundary conditions. The Fourier amplitude of the additional potential is given by the equation

$$A(\kappa, \theta) = \frac{1}{2\pi\kappa} \frac{(\kappa_0 + \kappa\cos^2\theta)}{(\kappa_0 - \kappa\cos^2\theta)} \tag{22}$$

The natural expansion into partial fractions is expressed by the equation

$$A(\kappa, \theta) = -\frac{1}{2\pi\kappa} + \frac{\kappa_0}{\pi\kappa(\kappa_0 - \kappa\cos^2\theta)} \tag{23}$$

The first term of the expansion is the amplitude of a negative image source. The potential $\varphi_2(x, y, z)$ for the image source is given by the equation

$$\varphi_2(x, y, z) = -\frac{1}{2\pi}\int_{-\pi}^{+\pi}\int_0^{\infty} e^{-\kappa(z+h)+i\kappa(x\cos\theta+y\sin\theta)}\,d\kappa\,d\theta \tag{24}$$

Thus the potential φ of the surface disturbance is given by the equation

$$\varphi(x, y, z) = \varphi_1(x, y, z) + \varphi_2(x, y, z) + \varphi_3(x, y, z) \tag{25}$$

where φ_1 is the potential of the source in an unbounded fluid, φ_2 is the potential of an image source over the free surface, and φ_3 is the potential of the surface wave train.

The potential φ_1 is given by the equation

$$\varphi_1 = \frac{1}{\{x^2 + y^2 + (z - h)^2\}^{\frac{1}{2}}} \tag{26}$$

Its derivatives are given by the equations

$$-\frac{\partial\varphi_1}{\partial x} = \frac{x}{\{x^2 + y^2 + (z - h)^2\}^{\frac{3}{2}}} \tag{27}$$

$$-\frac{\partial\varphi_1}{\partial y} = \frac{y}{\{x^2 + y^2 + (z - h)^2\}^{\frac{3}{2}}} \tag{28}$$

$$-\frac{\partial\varphi_1}{\partial z} = \frac{z - h}{\{x^2 + y^2 + (z - h)^2\}^{\frac{3}{2}}} \tag{29}$$

which are used explicity in the computation of flow from the point source.

The potential φ_2 is given by the equation

$$\varphi_2 = -\frac{1}{\{x^2 + y^2 + (z + h)^2\}^{\frac{1}{2}}} \tag{30}$$

Its derivatives are given by the equations

$$-\frac{\partial\varphi_2}{\partial x} = -\frac{x}{\{x^2 + y^2 + (z + h)^2\}^{\frac{3}{2}}} \tag{31}$$

$$-\frac{\partial\varphi_2}{\partial y} = -\frac{y}{\{x^2 + y^2 + (z + h)^2\}^{\frac{3}{2}}} \tag{32}$$

$$-\frac{\partial\varphi_2}{\partial z} = -\frac{z + h}{\{x^2 + y^2 + (z + h)^2\}^{\frac{3}{2}}} \tag{33}$$

which are used explicitly in the computation of flow from the image source.

The Fourier analysis of the steady state gives only the Cauchy principal value of a double integral. The full expression for the potential of the steady wave is derived in Appendix A from the limit which is approached by a transient wave.

The potential φ_3 is given by the equation

$$\varphi_3(x, y, z) = +i\kappa_0 \int_{-\pi}^{+\pi} \frac{\cos\theta}{|\cos^3\theta|} e^{-\frac{\kappa_0}{\cos^2\theta}(z+h)+\frac{i\kappa_0}{\cos^2\theta}(x\cos\theta+y\sin\theta)}\, d\theta$$

$$+ PV \frac{\kappa_0}{\pi} \int_{-\pi}^{+\pi} \int_0^{\infty} \frac{e^{-\kappa(z+h)+i\kappa(x\cos\theta+y\sin\theta)}}{\kappa_0 - \kappa\cos^2\theta}\, d\kappa\, d\theta \tag{34}$$

The integration can be simplified through the substitutions

$$\delta = \frac{\kappa_0}{\cos^2\theta}\left\{(z+h) - i(x\cos\theta + y\sin\theta)\right\} \tag{35}$$

and

$$\kappa = \frac{\kappa_0}{\cos^2\theta} - \frac{u}{(z+h) - i(x\cos\theta + y\sin\theta)} \tag{36}$$

where u is a new variable of integration. The function $e^{-\delta}$ itself satisfies both Laplace's equation and the free-surface boundary equation. It is added in just the right amount to make the wave train trail behind the source if the path of integration with respect to u proceeds along a straight line in the complex plane from $u = -\infty$ toward the origin, bypasses the origin on a half circle of small radius at the origin, and continues on a straight line to $u = \delta$. Whether the half circle is on the right or on the left of the origin depends upon the sign of $\cos\theta$. The path of integration in each case is illustrated in Figure 2. The potential of the surface wave is given by the equation

$$\varphi_3(x, y, z) = \frac{\kappa_0}{\pi} \int_{-\pi}^{+\pi} \frac{e^{-\delta}}{\cos^2\theta} \int_{-\infty}^{\delta} \frac{e^u}{u}\, du\, d\theta \tag{37}$$

Differentiation of this potential requires the equation

$$\frac{d}{d\delta} e^{-\delta} \int_{-\infty}^{\delta} \frac{e^u}{u}\, du = \frac{1}{\delta} - e^{-\delta} \int_{-\infty}^{\delta} \frac{e^u}{u}\, du \tag{38}$$

which may be integrated by parts to give the equation

$$\frac{d}{d\delta} e^{-\delta} \int_{-\infty}^{\delta} \frac{e^u}{u}\, du = -e^{-\delta} \int_{-\infty}^{\delta} \frac{e^u}{u^2}\, du \tag{39}$$

The derivatives of φ_3 are given by the equations

$$-\frac{\partial\varphi_3}{\partial x} = -i\frac{\kappa_0^2}{\pi} \int_{-\pi}^{+\pi} \frac{e^{-\delta}}{\cos^4\theta} \int_{-\infty}^{\delta} \frac{e^u}{u^2}\, du \cos\theta\, d\theta \tag{40}$$

$$-\frac{\partial\varphi_3}{\partial y} = -i\frac{\kappa_0^2}{\pi} \int_{-\pi}^{+\pi} \frac{e^{-\delta}}{\cos^4\theta} \int_{-\infty}^{\delta} \frac{e^u}{u^2}\, du \sin\theta\, d\theta \tag{41}$$

$$-\frac{\partial\varphi_3}{\partial z} = +\frac{\kappa_0^2}{\pi} \int_{-\pi}^{+\pi} \frac{e^{-\delta}}{\cos^4\theta} \int_{-\infty}^{\delta} \frac{e^u}{u^2}\, du\, d\theta \tag{42}$$

When θ is replaced in the integrands by $\theta \pm \pi$ the integrands are replaced by their complex conjugates. The real parts of integrals from $-\pi$ to $+\pi$ are equal to twice the real parts of integrals from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$.

It is convenient to introduce new parameters μ, ω which are defined by the equations

$$\mu = -(x + h) \qquad\qquad \omega = x\cos\theta + y\sin\theta \tag{43}$$

whence the parameter δ is given by the equation

$$\delta = -\frac{\kappa_0}{\cos^2\theta}(\mu + i\omega) \tag{44}$$

The parameter δ has always a positive real part, but it may have a positive or negative imaginary part according to the value of θ.

It is convenient to introduce a new parameter t which is defined by the equation

$$t = \tan\theta \tag{45}$$

The potential φ_3 is given by the equation

$$\varphi_3(x, y, z) = \frac{2\kappa_0}{\pi}\int_{-\infty}^{+\infty} e^{-\delta}\int_{-\infty}^{\delta}\frac{e^u}{u}\,du\,dt \tag{46}$$

and the derivatives of φ_3 are given by the equations

$$-\frac{\partial\varphi_3}{\partial x} = -i\,\frac{2\kappa_0^2}{\pi}\int_{-\infty}^{+\infty}\sqrt{1+t^2}\;e^{-\delta}\int_{-\infty}^{\delta}\frac{e^u}{u^2}\,du\,dt \tag{47}$$

$$-\frac{\partial\varphi_3}{\partial y} = -i\,\frac{2\kappa_0^2}{\pi}\int_{-\infty}^{+\infty} t\sqrt{1+t^2}\;e^{-\delta}\int_{-\infty}^{\delta}\frac{e^u}{u^2}\,du\,dt \tag{48}$$

$$-\frac{\partial\varphi_3}{\partial z} = +\,\frac{2\kappa_0^2}{\pi}\int_{-\infty}^{+\infty}(1+t^2)\;e^{-\delta}\int_{-\infty}^{\delta}\frac{e^u}{u^2}\,du\,dt \tag{49}$$

while the parameter δ is given by the equation

$$\delta = -\kappa_0\mu(1+t^2) - i\kappa_0(x+yt)\sqrt{1+t^2} \tag{50}$$

If the coordinates h, x, y, z are scaled through multiplication by κ_0 before computation, then the analysis may be simplified to the case for which $\kappa_0 = 1$.

COMPLEX PHASE

Stationary Phase

The complex phase δ is given in terms of t by the equation

$$\delta = -\mu(1+t^2) - i(x+yt)\sqrt{1+t^2} \tag{51}$$

There are two branches of the Riemann surface over the t-plane. The primary branch contains the real axis, on which the radical is positive. The secondary branch is reached along any path which encircles one of the branch points at $\pm i$. All members of the equation may be brought to the same side, and the expression so obtained may be cleared of the radical through multiplication by the same expression with the sign of the radical reversed. The result is the quartic equation

$$x^2 + (\mu+\delta)^2 + 2xyt + \{x^2 + y^2 + 2\mu(\mu+\delta)\}t^2 + 2xyt^3 + (y^2+\mu^2)t^4 = 0 \tag{52}$$

There are four branches of the Riemann surface over the δ-plane.

The complex phase δ is stationary where t satisfies the equation

$$\frac{d\delta}{dt} = -2\mu t - i\,\frac{(y + xt + 2yt^2)}{\sqrt{1+t^2}} = 0 \tag{53}$$

The terms of the equation may be transposed and squared to clear the equation of the radical. The result is the quartic equation

$$y^2 + 2xyt + (x^2 + 4y^2 + 4\mu^2)t^2 + 4xyt^3 + 4(y^2 + \mu^2)t^4 = 0 \tag{54}$$

which has real coefficients and has two complex conjugate pairs of roots. One pair of roots is associated with the primary branch and the other pair of roots is associated with the secondary branch. The roots of the quartic equation could be found by the Ferrari method, but better control over identification and accuracy is provided by Newton–Raphson iteration with the irrational equation.

The connection between the complex δ-plane and the complex t-plane is illustrated by Figures 3 and 4, where solid lines are various branches of the real axis in the t-plane, and dots and asterisks mark the points of stationary complex phase. In the δ-plane, the upper curve belongs to the primary branch, while the lower curve belongs to the secondary branch. In the t-plane, the two lines which cross the imaginary axis belong to the primary branch, while the two lines which bend back from the imaginary axis belong to the secondary branch.

Initial Approximation

Solution of the Equation $d\delta/dt = 0$ is represented in an Argand diagram by the vanishing of the sum of the three vectors

$$-2\mu it\sqrt{1+t^2} \qquad\qquad xt \qquad\qquad 2y(\tfrac{1}{2} + t^2) \tag{55}$$

These vectors cannot have a zero sum for a real value of t unless x and y both are zero. The first and second vectors cannot be collinear unless t is imaginary with absolute value greater than 1, in which case the third vector cannot be collinear with the other two. The three vectors can have a vanishing combination only if the first and third have components of opposite sign in the direction perpendicular to the second vector. The limit of possibility is where the second and third vectors are collinear, in which case t^2 and $\frac{1}{2}$ form two sides of a triangle while the third side lies in the direction of t. The angle which t^2 makes with the real axis is equal to twice that angle which t makes, either with the real axis, or with the side $\frac{1}{2}$. The triangle has two angles equal and is isosceles. The equal sides are t^2 and $\frac{1}{2}$, whence the limiting absolute value of t is $1/\sqrt{2}$.

If the absolute value of t is less than $1/\sqrt{2}$, then $\sqrt{1+t^2}$ deviates from the direction of the real axis by at most $\frac{1}{8}\pi$. The first vector (with negative μ) lies to the left of the direction of t, whereas the third vector (with positive y) lies on the positive real side of the direction of t. The first and third vectors oppose each other only if the imaginary part of t is positive.

If the absolute value of t is more than $1/\sqrt{2}$, then $\sqrt{1+t^2}$ still lies on the positive real side of the imaginary axis, provided the t-plane is cut outward along the imaginary axis from $\pm i$. The first vector still lies to the left of the direction of t, but the third vector now lies on the negative real side of the direction of t. The first and third vectors oppose each other only if the imaginary part of t is negative.

It may be concluded that the equation $d\delta/dt = 0$ has only two primary roots, of which one lies on the positive imaginary side of the real axis inside a circle of radius $1/\sqrt{2}$ while the other lies on the negative imaginary side of the real axis outside a circle of radius $1/\sqrt{2}$.

In the limit when $\mu = 0$ the equation $d\delta/dt = 0$ is simplified to the equation

$$\frac{1}{2} + \frac{x}{2y}t + t^2 = 0 \tag{56}$$

The roots of this equation are given by the equations

$$t_1 = \frac{-x - \sqrt{x^2 - 8y^2}}{4y} \qquad t_2 = \frac{-x + \sqrt{x^2 - 8y^2}}{4y} \tag{57}$$

The roots lie on the real axis when $|x| > \sqrt{8}|y|$, while the roots lie on the circle of radius $1/\sqrt{2}$ when $|x| < \sqrt{8}|y|$.

In the limit when $x = 0$ the quartic equation is simplified to the equation

$$\frac{y^2}{4(y^2 + \mu^2)} + t^2 + t^4 = 0 \tag{58}$$

The roots of this equation are given by the equations

$$t_1 = +\frac{i}{\sqrt{2}}\left[1 + \frac{\mu}{\sqrt{y^2 + \mu^2}}\right]^{\frac{1}{2}} \qquad t_2 = -\frac{i}{\sqrt{2}}\left[1 - \frac{\mu}{\sqrt{y^2 + \mu^2}}\right]^{\frac{1}{2}} \tag{59}$$

or by their complex conjugates. They are also the roots of the quadratic equation

$$\frac{\frac{1}{2}y}{\sqrt{y^2 + \mu^2}} + i\sqrt{1 - \frac{y}{\sqrt{y^2 + \mu^2}}}\; t + t^2 = 0 \tag{60}$$

which is derived from the polynomial $(t - t_1)(t - t_2)$.

In the limit when $y = 0$ the quartic equation is simplified to the equation

$$\frac{(x^2 + 4\mu^2)}{4\mu^2}\, t^2 + t^4 = 0 \tag{61}$$

The roots of this equation are given by the equations

$$t_1 = 0 \qquad t_2 = i\,\frac{\sqrt{x^2 + 4\mu^2}}{2\mu} \tag{62}$$

or by their complex conjugates. They are also the roots of the quadratic equation

$$-\frac{i}{2\mu}\sqrt{x^2 + 4\mu^2}\; t + t^2 = 0 \tag{63}$$

which is derived from the polynomial $(t - t_1)(t - t_2)$.

In the limit when $t \to 0$, the equation $d\delta/dt = 0$ is simplified to the equation

$$\frac{1}{2} + \frac{(ix + 2\mu)}{2iy}\, t + t^2 = 0 \tag{64}$$

which is derived from Equation (53) after neglect of the radical.

In the limit when $t \to \infty$, the equation $d\delta/dt = 0$ is simplified to the equation

$$\frac{1}{2} + \frac{xy - i\mu|x|}{2(y^2 + \mu^2)} t + t^2 = 0 \tag{65}$$

which is derived from Equation (53) after the radical has been replaced by the limiting approximation.

$$\sqrt{1 + t^2} \sim \mp t\left(1 + \frac{1}{2t^2}\right) \tag{66}$$

The sign is $\mp$ according to whether the sign of x is $\pm$.

The three special Equations (56), (60), (63) may be derived from the general equation

$$\frac{\frac{1}{2}y}{\sqrt{y^2 + \mu^2}} + \left[\frac{\frac{1}{2}xy - i(\mu^2 + \mu\sqrt{\frac{1}{4}x^2 + \mu^2})}{y^2 + \mu^2} + i\sqrt{1 - \frac{y}{\sqrt{y^2 + \mu^2}}} \right] t + t^2 = 0 \tag{67}$$

which is also compatible with the two limiting Equations (64), (65). The quadratic Equation (67) is synthetic, but it gives the correct roots on the real axis, on the imaginary axis, on the circle of radius $1/\sqrt{2}$, and it gives good enough roots elsewhere to serve as a good starting point for the Newton–Raphson iteration of the irrational Equation (53).

TRAPEZOIDAL INTEGRATION

In the straightforward evaluation of the Fourier integrals, the radial integration is expressed by the equation

$$e^{-\delta} \int_{-\infty}^{\delta} \frac{e^u}{u} \, du = e^{-\delta} Ei(\delta) \tag{68}$$

for which $Ei(\delta)$ is obtained by reference to a subroutine for the complex exponential integral. The subroutine gives the integral along a path which does not cross the positive real axis. A correction must be applied when the path of integration does cross the positive real axis. The correction

$$+ 2\pi i \, e^{-\delta} \tag{69}$$

is applied when $\mathfrak{Im}\,\delta > 0$. The radial integration is performed at equal intervals of the azimuth angle, and then the azimuthal integration is completed by application of a rule of quadrature.

In order to obtain information about amplitude and phase, the quadrature is applied through half a cycle of the azimuth angle. The real part is doubled and the imaginary part is cancelled when the quadrature is continued through a full cycle of the azimuth angle. Different quadrature rules are appropriate for the real part and the imaginary part.

A function which is continuous and periodic can be expressed in terms of its argument by a Fourier series. The coefficients in an infinite series are derived from integration of the products of the function and the sines or cosines of multiples of the argument. The coefficients in a finite series are derived from summations of the products of the function and the sines or cosines of multiples of equally spaced values of the argument. Orthogonality of the trigonometric terms is true for all orders in integration, but is not true for all orders in summation. If there are N discrete values of the argument then the summation of the products of sines or cosines is nonzero for orders whose sum is a multiple of N. Thus computed values of the coefficients for the finite series differ from the true values of coefficients for the infinite series as the result of an aliasing of the trigonometric terms. The number of coefficients in the finite series must be equal to the number of discrete arguments, and the terms of the finite series with sines and cosines are of order not higher than $\frac{1}{2}N$. The aliasing error in the constant of the finite series is of order at least as high as N.

All terms in either series except the constants vanish identically when either series is integrated through one cycle. The constant in the finite series is derived from the sum of the values of the function at the equally spaced values of the argument. Thus the rule for Fourier integration with respect to a cyclical variable is just the trapezoidal rule. An assessment of the accuracy of the trapezoidal rule requires an analysis of the aliasing errors. In the present case of Fourier integration the coefficients of the real part diminish almost exponentially with increasing order. The error of integration is determined almost entirely by the aliased term of lowest order. Because of the rapid decrease in the coefficients with increasing order the trapezoidal rule is the high accuracy rule for the real part. In the present case the coefficients of the imaginary part diminish with order almost in accordance with a quadratic polynomial in the reciprocal of the order. The aliasing error is half as large and reversed in sign when the values of the argument are shifted to midpoints between arguments. Inasmuch as the Simpson rule is equivalent to a superposition of two trapezoidal rules with shifted arguments, the aliasing errors almost cancel. The Simpson rule is a high accuracy rule for the imaginary part.

The accuracy of the integrations for the real part was confirmed by comparative computations in which the number of intervals of trapezoidal integration was increased in steps until the error was reduced to rounding error. The determination of error was repeated over enough coordinates in physical space to indicate the dependence of error upon position. An empirical formula for the estimation of the number of intervals for eight-digit accuracy is given by the equation

$$\frac{1}{4}N = 1 + \frac{27}{|\kappa_0\mu|^{\frac{1}{2}}} + \frac{\sqrt{x^2 + y^2}}{|\mu|}(7 + 54q + 7q^2) \tag{70}$$

where the argument q is given by the equation

$$q = \frac{y}{|\kappa_0\mu|^{\frac{1}{2}}\sqrt{x^2 + y^2}} \tag{71}$$

Inasmuch as the formula has only four variable terms, it can be made to satisfy a specification of accuracy at only four points. Almost everywhere else the level of accuracy should be better than the specified level.

The dependence of the error and the time on position is illustrated by dashed curves in Figures 5 and 6. Although the trapezoidal integration is not useful at small depth, it does provide data for checkouts at moderate depth.

ASYMPTOTIC APPROXIMATION

Exponential Integral

If $|\delta|$ is large throughout the integration with respect to θ, then the exponential integral is given by the asymptotic approximation

$$e^{-\delta}\int_{-\infty}^{\delta}\frac{e^{u}}{u}\,du \sim \sum_{n=0}^{N}\frac{n!}{\delta^{n+1}} \qquad (\mathcal{I}m\,\delta \le 0) \quad (72)$$

when the path of integration does not encircle the origin, but it is given by the approximation

$$e^{-\delta}\int_{-\infty}^{\delta}\frac{e^{u}}{u}\,du \sim \sum_{n=0}^{N}\frac{n!}{\delta^{n+1}} + 2\pi i\,e^{-\delta} \qquad (\mathcal{I}m\,\delta > 0) \quad (73)$$

when the path of integration does encircle the origin.

The terms with inverse powers of δ are monotonic while the term with $e^{-\delta}$ is oscillatory. The range of θ over which the oscillatory term should be included requires a determination of the value of θ for which $\mathcal{I}m\,\delta = 0$. Insofar as the exponential integral is an analytic function of θ, the path of integration in the δ-plane may be deformed. The point where the path of integration crosses the real axis may be displaced to a remote position on the real axis where the oscillatory term is negligible. The oscillatory term may be excluded for all θ when x is positive.

Monotonic Terms

The monotonic contribution to the potential φ_3 is given by the approximation

$$\varphi_3(x,y,z) \sim -\sum_{n=0}^{N}(-1)^n\frac{n!}{\pi}\oint\frac{\cos^{2n}\theta\,d\theta}{(\mu+i\omega)^{n+1}} \quad (74)$$

and the monotonic contributions to the derivatives of φ_3 are given by the approximations

$$-\frac{\partial\varphi_3}{\partial x} \sim i\sum_{n=1}^{N}(-1)^n\frac{n!}{\pi}\oint\frac{\cos^{2n-1}\theta\,d\theta}{(\mu+i\omega)^{n+1}} \quad (75)$$

$$-\frac{\partial\varphi_3}{\partial y} \sim i\sum_{n=1}^{N}(-1)^n\frac{n!}{\pi}\oint\frac{\cos^{2n-2}\theta\sin\theta\,d\theta}{(\mu+i\omega)^{n+1}} \quad (76)$$

$$-\frac{\partial\varphi_3}{\partial z} \sim -\sum_{n=1}^{N}(-1)^n\frac{n!}{\pi}\oint\frac{\cos^{2n-2}\theta\,d\theta}{(\mu+i\omega)^{n+1}} \quad (77)$$

The integrals of powers of $\mu + i\omega$ are computed and are stored in an array. The powers of $\cos\theta$ are expressed in terms of powers of $\mu + i\omega$. The results are combined to give the terms in the derivatives of φ_3.

Let r, ϕ be polar coordinates in physical space. The coordinates are related by the equations

$$x = r\cos\phi \qquad\qquad y = r\sin\phi \quad (78)$$

Then $\mu + i\omega$ is given by the equation

$$\mu + i\omega = \mu + i\sqrt{x^2 + y^2}\cos(\theta - \phi) \tag{79}$$

The integral of the reciprocal of $\mu + i\omega$ is given by Formula 300 on page 41 of Peirce's *Table of Integrals*[19]. The basic integrals of powers of $\mu + i\omega$ are given by the equations

$$\frac{1}{2\pi}\oint d\theta = 1 \tag{80}$$

$$\frac{\sqrt{x^2 + y^2}}{2\pi}\oint \frac{d\theta}{\mu + i\omega} = -\frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + \mu^2}} \tag{81}$$

A minus sign is required for the second integral because μ is negative. Integrals of higher and lower order can be generated through integration and differentiation.

Let the parameter u be defined by the equation

$$u = \frac{\mu}{\sqrt{x^2 + y^2}} \tag{82}$$

and let the integral $I_n(u)$ be defined by the equation

$$I_n(u) = \frac{1}{2\pi}\oint \frac{(\mu + i\omega)^n}{(x^2 + y^2)^{\frac{n}{2}}}\, d\theta \tag{83}$$

If n is positive, the integrals are generated by the recurrence equation

$$I_{n+1}(u) = (n+1)\int_0^u I_n(t)\,dt + \begin{cases} 0 & (n \text{ even}) \\ \dfrac{(-1)^{\frac{n+1}{2}}(n+1)!}{2^{n+1}(\frac{n+1}{2})!^2} & (n \text{ odd}) \end{cases} \tag{84}$$

where the constant of integration is derived from the beta function, and is nonzero only when n is odd. Each integral is represented by a power polynomial, for which the polynomial of lowest degree is given by Equation (80). The polynomials are integrated term by term with the aid of Equation (3) in Appendix B. If n is negative, the integrals are generated by the recurrence equation

$$I_{n-1}(u) = \frac{1}{n}\frac{d}{du} I_n(u) \tag{85}$$

Each integral is represented by the quotient between a power polynomial and a radical, for which the quotient of highest degree is given by Equation (81). The quotients are differentiated term by term with the aid of Equation (6) in Appendix B.

The trigonometric functions of θ are given by the identities

$$\sin\theta = \sin\phi\cos(\theta-\phi) + \cos\phi\sin(\theta-\phi) \tag{86}$$

$$\cos\theta = \cos\phi\cos(\theta-\phi) - \sin\phi\sin(\theta-\phi) \tag{87}$$

Let $P_n(\theta)$ and $Q_n(\theta)$ be functions such that

$$\cos^n\theta = P_n(\theta) + \sin(\theta-\phi)Q_n(\theta) \tag{88}$$

The functions of lowest order are given by the equations

$$P_0(\theta) = 1 \qquad\qquad Q_0(\theta) = 0 \tag{89}$$

In view of the equation

$$\cos^{n+1}\theta = \cos\theta\,\{P_n(\theta) + \sin(\theta-\phi)Q_n(\theta)\} \tag{90}$$

it follows that the functions are related by the recurrence equations

$$P_{n+1}(\theta) = \cos\phi\cos(\theta-\phi)P_n(\theta) - \sin\phi\sin^2(\theta-\phi)Q_n(\theta) \tag{91}$$

$$Q_{n+1}(\theta) = -\sin\phi\,P_n(\theta) + \cos\phi\cos(\theta-\phi)Q_n(\theta) \tag{92}$$

The functions $P_n(\theta)$ and $Q_n(\theta)$ thus are constructed as power polynomials in $\cos(\theta-\phi)$.
Let $P'_n(\theta)$ and $Q'_n(\theta)$ be functions such that

$$\cos^n\theta\sin\theta = P'_n(\theta) + \sin(\theta-\phi)Q'_n(\theta) \tag{93}$$

The functions of lowest order are given by the equations

$$P'_0(\theta) = \sin\phi\cos(\theta-\phi) \qquad\qquad Q'_0(\theta) = \cos\phi \tag{94}$$

In view of the equation

$$\cos^n\theta\sin\theta = \sin\theta\,\{P_n(\theta) + \sin(\theta-\phi)Q_n(\theta)\} \tag{95}$$

it follows that the functions are related by the recurrence equations

$$P'_n(\theta) = \sin\phi\cos(\theta-\phi)P_n(\theta) + \cos\phi\sin^2(\theta-\phi)Q_n(\theta) \tag{96}$$

$$Q'_n(\theta) = \cos\phi\,P_n(\theta) + \sin\phi\cos(\theta-\phi)Q_n(\theta) \tag{97}$$

The functions $P'_n(\theta)$ and $Q'_n(\theta)$ thus are constructed as power polynomials in $\cos(\theta-\phi)$.
The trigonometric functions of $\theta-\phi$ are given by the equations

$$\cos(\theta-\phi) = \frac{i\mu - i(\mu+i\omega)}{\sqrt{x^2+y^2}} \tag{98}$$

$$\sin^2(\theta-\phi) = \frac{x^2+y^2+\mu^2-2\mu(\mu+i\omega)+(\mu+i\omega)^2}{x^2+y^2} \tag{99}$$

Substitutions in the recurrence equations lead to the expression of the trigonometric functions of θ as power polynomials in $\mu+i\omega$.

Oscillatory Terms

The oscillatory contribution to the potential φ_3 is given by the approximation

$$\varphi_3(x, y, z) \sim 4i \int_{-\infty}^{+\infty} e^{-\delta}\, dt \qquad (100)$$

for which

$$1 \qquad (101)$$

is a monotonic factor of the integrand, and the oscillatory contributions to the derivatives of φ_3 are given by the approximations

$$-\frac{\partial \varphi_3}{\partial x} \sim 4 \int_{-\infty}^{+\infty} \sqrt{1+t^2}\; e^{-\delta}\, dt \qquad (102)$$

$$-\frac{\partial \varphi_3}{\partial y} \sim 4 \int_{-\infty}^{+\infty} t\sqrt{1+t^2}\; e^{-\delta}\, dt \qquad (103)$$

$$-\frac{\partial \varphi_3}{\partial z} \sim 4i \int_{-\infty}^{+\infty} (1+t^2)\; e^{-\delta}\, dt \qquad (104)$$

for which

$$\sqrt{1+t^2} \qquad t\sqrt{1+t^2} \qquad 1+t^2 \qquad (105)$$

are monotonic factors of the integrands.

Required for Taylor series expansions of the monotonic factors are derivatives which can be expressed as the quotients between power polynomials in t and powers of $\sqrt{1+t^2}$. The algorithm for generating successive derivatives of the quotients is given by Equation (6) in Appendix B.

The series converge in the t-plane only within that circle which is centered at the center of expansion and passes through the nearest singularity at $\pm i$. When each series of limited convergence is multiplied by an exponential function of its argument and the product is integrated from $-\infty$ to $+\infty$, the series becomes asymptotic.

Evaluation of the integrals can be completed if the variable t is replaced by a new variable u such that the parameter δ is a polynomial of at most the third degree in the variable u.

The parameter δ and its derivatives are given in terms of t by the equations

$$\delta = -\mu(1+t^2) - i(x+yt)\sqrt{1+t^2} \qquad (106)$$

$$\frac{d\delta}{dt} = -2\mu t - i\frac{(y + xt + 2yt^2)}{\sqrt{1+t^2}} \qquad (107)$$

$$\frac{d^2\delta}{dt^2} = -2\mu - i\frac{(x + 3yt + 2yt^3)}{(1+t^2)^{\frac{3}{2}}} \qquad (108)$$

The parameter δ and its derivatives are given in terms of u by the equations

$$\delta = \delta_0 + \delta_0' u + \frac{\delta_0'' u^2}{2!} + \frac{\delta_0''' u^3}{3!} \tag{109}$$

$$\frac{d\delta}{du} = \delta_0' + \delta_0'' u + \frac{\delta_0''' u^2}{2!} \tag{110}$$

$$\frac{d^2\delta}{du^2} = \delta_0'' + \delta_0''' u \tag{111}$$

Equivalence of the expressions defines a series expansion of t in terms of u provided $d\delta/du = 0$ at the same value of δ as $d\delta/dt = 0$. Otherwise dt/du would be infinite where $d\delta/dt = 0$. The equation $d\delta/dt = 0$ has roots where the values of t are t_1 and t_2 and the values of δ are δ_1 and δ_2.

Near the centerline behind the source the points of stationary phase are far apart. The path of integration in the t-plane may be deformed to pass through the points t_1 and t_2 of stationary phase. The contribution to integration is large near t_1 and t_2, and is small between t_1 and t_2. Each integration from t_1 or t_2 to an intermediate point may be approximated by an integration to $\pm\infty$.

The polynomial for δ is simplified by setting $\delta_0''' = 0$. The conventional substitution

$$u \to u - \frac{\delta_0'}{\delta_0''} \tag{112}$$

eliminates also δ_0'. The constant δ_0 is given by the substitution

$$\delta_0 \to \delta \tag{113}$$

and the second coefficient δ_0'' is given by the substitution

$$\delta_0'' \to \frac{d^2\delta}{dt^2} \tag{114}$$

for which t has the values t_1 or t_2 where $d\delta/dt = 0$. The algorithm for converting the variable of integration from t to u is given by Equation (15) in Appendix B.

Integration of $e^{-\delta}$ with respect to u is given in terms of exponential functions by the equation

$$\int_{-\infty}^{+\infty} e^{-\delta}\, du = \left(\frac{2\pi}{\delta_0''}\right)^{\frac{1}{2}} e^{-\delta_0} \tag{115}$$

Integration of the product of u and $e^{-\delta}$ is zero by symmetry. Integration of the product of a power of u and $e^{-\delta}$ is given by the recurrence

$$\int_{-\infty}^{+\infty} u^{2n} e^{-\delta}\, du = -2\frac{d}{d\delta_0''}\int_{-\infty}^{+\infty} u^{2n-2} e^{-\delta}\, du \tag{116}$$

or after iteration by the equation

$$\int_{-\infty}^{+\infty} u^{2n} e^{-\delta}\, du = \frac{(2n)!(2\pi)^{\frac{1}{2}}}{2^n n!(\delta_0'')^{n+\frac{1}{2}}} e^{-\delta_0} \tag{117}$$

Insofar as integration gives monotonic terms, summation can be continued until a new term is larger than the preceding term.

Along the line of wave crests the points of stationary phase are close together. The path of integration in the t-plane may be deformed to pass midway between the points t_1 and t_2 of stationary phase.

The quadratic term in the cubic polynomial can be eliminated by the conventional transformation

$$u \to u - \frac{\delta_0''}{\delta_0'''} \tag{118}$$

The variable u thus can be redefined so that $\delta_0'' = 0$. Inasmuch as u is a new variable, it may be scaled arbitrarily. Let it be scaled so that $\delta_0''' = -2i$. If the limits of integration lie above the complex value

$$\pm N + \frac{|\mathcal{R}e\,\delta_0'|}{N}\,i \tag{119}$$

then the real part of $-\delta$ goes to $-\infty$ in the limit as $N \to \infty$. The integral of $e^{-\delta}$ with respect to u is guaranteed to converge if the path of integration lies just above the real axis.

The equation $d\delta/du = 0$ is . mplified to the equation

$$u^2 = -i\delta_0' \tag{120}$$

for which the value of δ is given by the expression

$$\delta_0 \pm \tfrac{2}{3}(i\delta_0')^{\frac{3}{2}} \tag{121}$$

The values of δ_0 and δ_0' are given in terms of the values of δ_1 and δ_2 by the equations

$$\delta_0 = \tfrac{1}{2}(\delta_1 + \delta_2) \tag{122}$$

$$\delta_0' = -i\{\tfrac{3}{4}(\delta_1 - \delta_2)\}^{\frac{2}{3}} \tag{123}$$

The value of t for which $\delta = \delta_0$ is determined by Newton–Raphson iteration. The value of t at which the iteration is started is given by the approximation $t \sim \frac{1}{2}(t_1 + t_2)$. The algorithm for converting the variable of integration from t to u is given by Equation (17) in Appendix B.

Integration of $e^{-\delta}$ with respect to u is given in terms of Airy functions by the equation

$$\int_{-\infty}^{+\infty} e^{-\delta}\,du = 2\pi e^{-\delta_0} Ai(i\delta_0') \tag{124}$$

Integration of the product of a power of u and $e^{-\delta}$ is given by the equation

$$\int_{-\infty}^{+\infty} u^n e^{-\delta}\,du = 2\pi e^{-\delta_0}(-i)^n Ai^{(n)}(i\delta_0') \tag{125}$$

where $Ai^{(n)}(i\delta_0')$ is the nth derivative of $Ai(i\delta_0')$. The algorithm for generating the successive derivatives is given by Equations (26) and (27) in Appendix B.

Inasmuch as integration gives terms which alternate between two series of values, summation is continued until a new term is larger than both of the two previous terms or until the terms make no change in the sum. In a range of arguments the terms increase at first before they decrease, and finally increase. In order to keep the initial increase from triggering a premature termination of the series, the first three terms are evaluated prior to the start of sensing for termination of the series.

Accuracy and Efficiency

Transition from the quadratic approximation to the cubic approximation is placed where the errors of computation are the same for both approximations. The cubic approximation is used where the coordinates satisfy the inequality

$$|x| > 6|y| \tag{126}$$

Transition from truncation error to rounding error occurs at a critical radius from the source. The limiting radius for acceptable truncation error varies with direction. The critical radius is given by the empirical equation

$$\kappa_0\sqrt{x^2 + y^2} = 27 + 21\frac{y^2}{x^2 + y^2} \tag{127}$$

The rounding error for radii larger than the limiting radius also varies with direction. The rounding error along the critical line of wave crests increases with decrease in the depth below the free surface. The relatively large errors on the critical line are attributed to the close approach to each other of the points of stationary phase. Thus a perturbation of the position of the points of stationary phase causes a significant change in the error in velocity. The error is tolerated insofar as the asymptotic approximation is faster than the integration by parts. Errors and times are illustrated by dotted curves in Figures 5 and 6.

INTEGRATION BY PARTS

Interpolation

In the integration with respect to t the integrands are the products of the monotonic factor

$$1 \tag{128}$$

and the oscillatory factor

$$e^{-t}\int_{-\infty}^{t}\frac{e^u}{u}\,du \tag{129}$$

or the integrands are the products of the monotonic factors

$$\sqrt{1 + t^2} \qquad t\sqrt{1 + t^2} \qquad 1 + t^2 \tag{130}$$

and the oscillatory factor

$$e^{-t}\int_{-\infty}^{t}\frac{e^u}{u^2}\,du \tag{131}$$

If the monotonic factors and the oscillatory factors are expressed in terms of a common argument, then the integration can be completed through integrations by parts. The parameter δ is expressed by the equation

$$\delta = \eta + \epsilon \tag{132}$$

where η is a center of expansion and ϵ is the common argument of the monotonic factors and the oscillatory factors.

The real part of η always is positive. The real part of $\eta^{1/2}$ always is negative, because η is on the same branch of the Riemann surface as $-\infty$. The sign of $\epsilon^{1/2}$ is arbitrary. The sign is adjusted so as to make the phase angle of $\epsilon^{1/2}$ equal to half the phase angle of ϵ.

Subtraction of η leads to the equation

$$\epsilon = -(\eta + \mu) - \mu t^2 - i(x + yt)\sqrt{1 + t^2} \tag{133}$$

and division by t^2 leads to the equation

$$\frac{\epsilon}{t^2} = -\mu - \frac{\eta + \mu}{t^2} - i\left(y + \frac{x}{t}\right)\sqrt{1 + \frac{1}{t^2}} \tag{134}$$

Taylor series expansions express ϵ directly in terms of t, then transformations of variable express t conversely in terms of ϵ. The algorithms for the series expansions and the transformations of variable are the basis for a previous subroutine which has been described in a previous report. This subroutine spent too much time in processing the Taylor series expansions. A gain in speed by a factor of seven has been achieved when the Taylor series expansions have been replaced by Lagrange interpolation. The algorithms for the Lagrange interpolation are the basis for the current subroutine which is described in the present report.

Series expansions of the monotonic factors are derived by Lagrange interpolation between eleven discrete values of the monotonic factors. The discrete values are computed at each discrete argument in a set whose spacing between arguments is proportional to the spacing between the roots of a Chebyshev polynomial. The accuracy of approximation between discrete arguments then tends to be uniform in accordance with the Chebyshev criterion. Where the path of integration is curved, the spacing along the curve is approximated by chords which span segments of the curve. The range of approximation is determined by the configuration in the δ−plane, while the path of integration is specified in the t−plane. The position in the t−plane for a specified chord in the δ−plane is derived by Newton−Raphson iteration.

The natural path of integration is the real axis in the t−plane, but this path passes between two singularities where $d\delta/dt = 0$. Many intervals of integration would be required in the vicinity of the singularities. These can be reduced to a single interval near the first singularity, and they can be reduced to fewer intervals near the second singularity, if the path of integration is displaced to pass right through the first singularity. The integration is completed in a single final interval.

In the integrations with respect to ϵ the monotonic factors are

$$\frac{1}{\dfrac{d\delta}{dt}} \tag{135}$$

and

$$\frac{\sqrt{1+t^2}}{\dfrac{d\delta}{dt}} \qquad \frac{t\sqrt{1+t^2}}{\dfrac{d\delta}{dt}} \qquad \frac{(1+t^2)}{\dfrac{d\delta}{dt}} \tag{136}$$

Different expansions are required for each interval of integration.

In the first interval of integration, the center of expansion η is located at that value of δ where $t = t_1$, and $d\delta/dt = 0$. The parameters

$$\frac{\epsilon^{\frac{1}{2}}}{\dfrac{d\delta}{dt}} \tag{137}$$

and

$$\frac{\epsilon^{\frac{1}{2}}\sqrt{1+t^2}}{\dfrac{d\delta}{dt}} \qquad \frac{\epsilon^{\frac{1}{2}}t\sqrt{1+t^2}}{\dfrac{d\delta}{dt}} \qquad \frac{\epsilon^{\frac{1}{2}}(1+t^2)}{\dfrac{d\delta}{dt}} \tag{138}$$

are computed for a discrete set of values of $\epsilon^{1/2}$. Their limiting values at $\epsilon = 0$ are

$$\frac{1}{\sqrt{2}\left(\dfrac{d^2\delta}{dt^2}\right)^{\frac{1}{2}}} \tag{139}$$

and

$$\frac{\sqrt{1+t_1^2}}{\sqrt{2}\left(\dfrac{d^2\delta}{dt^2}\right)^{\frac{1}{2}}} \qquad \frac{t_1\sqrt{1+t_1^2}}{\sqrt{2}\left(\dfrac{d^2\delta}{dt^2}\right)^{\frac{1}{2}}} \qquad \frac{(1+t_1^2)}{\sqrt{2}\left(\dfrac{d^2\delta}{dt^2}\right)^{\frac{1}{2}}} \tag{140}$$

Expansion in series leads to the representation of the monotonic factors as ascending series in powers of $\epsilon^{1/2}$ which begin with $\epsilon^{-1/2}$.

In the continuation of integration, the center of expansion η is located at the center of the range of approximation. The monotonic factors are expressed directly as series in powers of ϵ.

In the final interval of integration, the center of expansion η is located at that value of δ where $d\delta/dt = 0$ and δ is nearest to $+\infty$. The parameters

$$\frac{\epsilon^{\frac{1}{2}}}{\dfrac{d\delta}{dt}} \tag{141}$$

and

$$\frac{\sqrt{1+t^2}}{\epsilon^{\frac{1}{2}}\dfrac{d\delta}{dt}} \qquad \frac{t\sqrt{1+t^2}}{\epsilon^{\frac{1}{2}}\dfrac{d\delta}{dt}} \qquad \frac{(1+t^2)}{\epsilon^{\frac{1}{2}}\dfrac{d\delta}{dt}} \tag{142}$$

are computed for a discrete set of values of $\epsilon^{-1/2}$. Their limiting values at $\epsilon = \infty$ are

$$\frac{\pm\frac{1}{2}}{(-\mu \mp iy)^{\frac{1}{2}}} \tag{143}$$

and

$$0 \qquad\qquad \frac{\frac{1}{2}}{(-\mu \mp iy)^{\frac{3}{2}}} \qquad\qquad \frac{\pm\frac{1}{2}}{(-\mu \mp iy)^{\frac{3}{2}}} \tag{144}$$

where the $\pm$ signs are determined according to whether the direction of integration is $\pm$ in the t-plane. Expansion in series leads to the representation of the monotonic factors as descending series in powers of $\epsilon^{-1/2}$ which begin with $\epsilon^{+1/2}$.

The range of expansion of the Lagrange interpolation is subject to limitations which are similar to the limitations on the range of expansion of the Taylor series. The expansions are limited to the primary branch in the δ-plane as long as the radius of expansion is less than the distance from the center of expansion to the points where $\delta = 0$ and $t = \pm i$. The expansions are convergent if the radius of expansion also is less than the distance from the center of expansion to the nearest point in the primary branch where $d\delta/dt = 0$ The actual range of approximation is a fraction of the radius of convergence. The value of the fraction is determined by computation. In the initial interval of integration the fraction for expansion in terms of $\epsilon^{1/2}$ is one half, in the continuation of integration the fraction for expansion in terms of ϵ is two thirds, and in the final interval of integration the fraction for expansion in terms of $\epsilon^{-1/2}$ is one half. Boundaries of approximation and convergence are illustrated by dotted lines in Figures 3 and 4 for the initial interval of integration.

During the stepping from one interval to the next interval the center of expansion for the new interval is estimated on the assumption that the new interval is a linear extension of the old interval. The position of the center of the new interval is estimated by an application of the cosine law to a triangle with vertices at the old center of expansion, the nearest singularity, and the new center of expansion.

Recurrence

The terms of the ascending series are integrated with the aid of the recurrence equation

$$\int_0^{\epsilon} u^n e^{-(\eta+u)} \int_{-\infty}^{\eta+u} \frac{e^t}{t}\, dt\, du = \int_0^{\epsilon} \frac{u^n}{\eta+u}\, du - \epsilon^n e^{-(\eta+\epsilon)} \int_{-\infty}^{\eta+\epsilon} \frac{e^t}{t}\, dt$$
$$+ n \int_0^{\epsilon} u^{n-1} e^{-(\eta+u)} \int_{-\infty}^{\eta+u} \frac{e^t}{t}\, dt\, du \tag{145}$$

which is started with the aid of the special equations

$$\int_0^{\epsilon} \frac{e^{-(\eta+u)}}{u^{\frac{1}{2}}} \int_{-\infty}^{\eta+u} \frac{e^t}{t}\, dt\, du = 2e^{-\eta} \int_{-\infty}^{\eta} \frac{e^t}{t^{\frac{1}{2}}} \tan^{-1}\left(\frac{\epsilon}{t}\right)^{\frac{1}{2}} dt \tag{146}$$

$$\int_0^{\epsilon} e^{-(\eta+u)} \int_{-\infty}^{\eta+u} \frac{e^t}{t}\, dt\, du = \log\left(1 + \frac{\epsilon}{\eta}\right) + e^{-\eta} \int_{-\infty}^{\eta} \frac{e^t}{t}\, dt - e^{-(\eta+\epsilon)} \int_{-\infty}^{\eta+\epsilon} \frac{e^t}{t}\, dt \tag{147}$$

and with the aid of the recurrence equation

$$\int_0^\epsilon u^n e^{-(\eta+u)} \int_{-\infty}^{\eta+u} \frac{e^t}{t^2}\, dt\, du = n \int_0^\epsilon \frac{u^{n-1}}{\eta+u}\, du - \epsilon^n e^{-(\eta+\epsilon)} \int_{-\infty}^{\eta+\epsilon} \frac{e^t}{t}\, dt$$

$$+ n \int_0^\epsilon u^{n-1} e^{-(\eta+u)} \int_{-\infty}^{\eta+u} \frac{e^t}{t^2}\, dt\, du \tag{148}$$

which is started with the aid of the special equations

$$\int_0^\epsilon \frac{e^{-(\eta+u)}}{u^{\frac{1}{2}}} \int_{-\infty}^{\eta+u} \frac{e^t}{t^2}\, dt\, du = -\frac{2}{\eta^{\frac{1}{2}}} \tan^{-1}\left(\frac{\epsilon}{\eta}\right)^{\frac{1}{2}} + 2e^{-\eta} \int_{-\infty}^{\eta} \frac{e^t}{t^{\frac{1}{2}}} \tan^{-1}\left(\frac{\epsilon}{t}\right)^{\frac{1}{2}} dt \tag{149}$$

$$\int_0^\epsilon e^{-(\eta+u)} \int_{-\infty}^{\eta+u} \frac{e^t}{t^2}\, dt\, du = e^{-\eta} \int_{-\infty}^{\eta} \frac{e^t}{t}\, dt \; - \; e^{-(\eta+\epsilon)} \int_{-\infty}^{\eta+\epsilon} \frac{e^t}{t}\, dt \tag{150}$$

The recurrence is cycled in the reverse direction when $|\epsilon| \leq 7$.

Required for ascending series are integrals which are generated by the recurrence equation

$$\int_0^\epsilon \frac{u^n}{\eta+u}\, du = \frac{\epsilon^n}{n} - \eta \int_0^\epsilon \frac{u^{n-1}}{\eta+u}\, du \tag{151}$$

The recurrence is started with the special integrals

$$\int_0^\epsilon \frac{du}{u^{\frac{1}{2}}(\eta+u)} = \frac{2}{\eta^{\frac{1}{2}}} \tan^{-1}\left(\frac{\epsilon}{\eta}\right)^{\frac{1}{2}} \tag{152}$$

$$\int_0^\epsilon \frac{du}{\eta+u} = \log\left(1 + \frac{\epsilon}{\eta}\right) \tag{153}$$

The integrals are given by the series expansion

$$\int_0^\epsilon \frac{u^n}{\eta+u}\, du = \eta^n \sum_{k=0}^{\infty} \frac{(-1)^k}{n+k+1} \left(\frac{\epsilon}{\eta}\right)^{n+k+1} \qquad (|\epsilon| < |\eta|) \tag{154}$$

The recurrence (151) removes progressively the terms of lowest order from the expansion, whence the residue does not retain the terms of highest order with enough accuracy when $|\epsilon| << |\eta|$.

The expansion (154) is generated in the direction of descending order by the recurrence

$$\int_0^\epsilon \frac{u^{n-1}}{\eta+u}\, du = \frac{1}{\eta}\left[\frac{\epsilon^n}{n} - \int_0^\epsilon \frac{u^n}{\eta+u}\, du\right] \tag{155}$$

which may be started with the limiting approximation

$$\int_0^\epsilon \frac{u^N}{\eta+u}\, du \sim \frac{\epsilon^{N+1}}{(N+1)(\eta+\epsilon)} \qquad (N \to \infty) \tag{156}$$

In view of the convergence condition for the series expansion, the recurrence (155) can be used only when $|\epsilon| < |\eta|$.

The terms of the descending series are integrated with the aid of the recurrence equation

$$\int_\epsilon^\infty u^{-n-1}e^{-(\eta+u)}\int_{-\infty}^{\eta+u}\frac{e^t}{t}\,dt\,du = \frac{1}{n}\int_\epsilon^\infty \frac{u^{-n}}{\eta+u}\,du + \frac{1}{n\epsilon^n}\,e^{-(\eta+\epsilon)}\int_{-\infty}^{\eta+\epsilon}\frac{e^t}{t}\,dt$$

$$-\frac{1}{n}\int_\epsilon^\infty u^{-n}e^{-(\eta+u)}\int_{-\infty}^{\eta+u}\frac{e^t}{t}\,dt\,du \tag{157}$$

which is started with the aid of the special equations

$$\int_\epsilon^\infty \frac{e^{-(\eta+u)}}{u^{\frac{1}{2}}}\int_{-\infty}^{\eta+u}\frac{e^t}{t}\,dt\,du = 2e^{-\eta}\int_{-\infty}^{\eta}\frac{e^t}{t^{\frac{1}{2}}}\tan^{-1}\left(\frac{t}{\epsilon}\right)^{\frac{1}{2}}dt \tag{158}$$

$$\int_\epsilon^\infty \frac{e^{-(\eta+u)}}{u}\int_{-\infty}^{\eta+u}\frac{e^t}{t}\,dt\,du = e^{-\eta}\int_{-\infty}^{\eta}\frac{e^t}{t}\log\left(1+\frac{t}{\epsilon}\right)dt \tag{159}$$

and with the aid of the recurrence equation

$$\int_\epsilon^\infty u^{-n-1}e^{-(\eta+u)}\int_{-\infty}^{\eta+u}\frac{e^t}{t^2}\,dt\,du = -\int_\epsilon^\infty \frac{u^{-n-1}}{\eta+u}\,du + \frac{1}{n\epsilon^n}\,e^{-(\eta+\epsilon)}\int_{-\infty}^{\eta+\epsilon}\frac{e^t}{t}\,dt$$

$$-\frac{1}{n}\int_\epsilon^\infty u^{-n}e^{-(\eta+u)}\int_{-\infty}^{\eta+u}\frac{e^t}{t^2}\,dt\,du \tag{160}$$

which is started with the aid of the special equations

$$\int_\epsilon^\infty u^{\frac{1}{2}}e^{-(\eta+u)}\int_{-\infty}^{\eta+u}\frac{e^t}{t^2}\,dt\,du = \epsilon^{\frac{1}{2}}e^{-(\eta+\epsilon)}\int_{-\infty}^{\eta+\epsilon}\frac{e^t}{t}\,dt + e^{-\eta}\int_{-\infty}^{\eta}\frac{e^t}{t^{\frac{1}{2}}}\tan^{-1}\left(\frac{t}{\epsilon}\right)^{\frac{1}{2}}dt \tag{161}$$

$$\int_\epsilon^\infty e^{-(\eta+u)}\int_{-\infty}^{\eta+u}\frac{e^t}{t^2}\,dt\,du = e^{-(\eta+\epsilon)}\int_{-\infty}^{\eta+\epsilon}\frac{e^t}{t}\,dt \tag{162}$$

$$\int_\epsilon^\infty \frac{e^{-(\eta+u)}}{u^{\frac{1}{2}}}\int_{-\infty}^{\eta+u}\frac{e^t}{t^2}\,dt\,du = -\frac{2}{\eta^{\frac{1}{2}}}\tan^{-1}\left(\frac{\eta}{\epsilon}\right)^{\frac{1}{2}} + 2e^{-\eta}\int_{-\infty}^{\eta}\frac{e^t}{t^{\frac{1}{2}}}\tan^{-1}\left(\frac{t}{\epsilon}\right)^{\frac{1}{2}}dt \tag{163}$$

$$\int_\epsilon^\infty \frac{e^{-(\eta+u)}}{u}\int_{-\infty}^{\eta+u}\frac{e^t}{t^2}\,dt\,du = -\frac{1}{\eta}\log\left(1+\frac{\eta}{\epsilon}\right) + e^{-\eta}\int_{-\infty}^{\eta}\frac{e^t}{t}\log\left(1+\frac{t}{\epsilon}\right)dt \tag{164}$$

The recurrence is cycled in the reverse direction when $|\epsilon| \geq 27$.

Required for descending series are integrals which are generated by the recurrence equation

$$\int_\epsilon^\infty \frac{u^{-n-1}}{\eta+u}\,du = \frac{1}{\eta}\left[\frac{1}{n\epsilon^n} - \int_\epsilon^\infty \frac{u^{-n}}{\eta+u}\,du\right] \tag{165}$$

The recurrence is started with the special integrals

$$\int_\epsilon^\infty \frac{du}{u^{\frac{1}{2}}(\eta+u)} = \frac{2}{\eta^{\frac{1}{2}}}\tan^{-1}\left(\frac{\eta}{\epsilon}\right)^{\frac{1}{2}} \tag{166}$$

$$\int_\epsilon^\infty \frac{du}{u(\eta+u)} = \frac{1}{\eta}\log\left(1+\frac{\eta}{\epsilon}\right) \tag{167}$$

The integrals are given by the series expansion

$$\int_{\epsilon}^{\infty} \frac{u^{-n}}{\eta + u}\, du = \frac{1}{\eta^{n}} \sum_{k=0}^{\infty} \frac{(-1)^{k}}{n+k}\left(\frac{\eta}{\epsilon}\right)^{n+k} \qquad (|\eta| < |\epsilon|) \quad (168)$$

The recurrence (165) removes progressively the terms of highest order from the expansion, whence the residue does not retain the terms of lowest order with enough accuracy when $|\eta| << |\epsilon|$

The expansion (168) is generated in the direction of ascending order by the recurrence

$$\int_{\epsilon}^{\infty} \frac{u^{-n}}{\eta + u}\, du = \frac{1}{n\epsilon^{n}} - \eta \int_{\epsilon}^{\infty} \frac{u^{-n-1}}{\eta + u}\, du \quad (169)$$

which may be started with the limiting approximation

$$\int_{\epsilon}^{\infty} \frac{u^{-N}}{\eta + u}\, du \sim \frac{\epsilon^{-N+1}}{(N-1)(\eta + \epsilon)} \qquad (N \to \infty) \quad (170)$$

In view of the convergence condition for the series expansion, the recurrence (169) can be used only when $|\eta| < |\epsilon|$.

The complex exponential integral

$$e^{-\eta} \int_{-\infty}^{\eta} \frac{e^{t}}{t}\, dt \quad (171)$$

and the complex Fresnel integral

$$e^{-\eta} \int_{-\infty}^{\eta} \frac{e^{t}}{t^{\frac{1}{2}}}\, dt \quad (172)$$

are obtained from subroutines[24]. It is necessary to apply corrections to bring the path of integration below each singularity at $t = 0$. The corrections are applied when

$$\mathcal{I}m\, \eta > 0 \quad (173)$$

For the exponential integral the correction is

$$+ 2\pi i\, e^{-\eta} \quad (174)$$

but for the Fresnel integral the correction is

$$- 2\sqrt{\pi}\, i\, e^{-\eta} - 2e^{-\eta} \int_{-\infty}^{\eta} \frac{e^{t}}{t^{\frac{1}{2}}}\, dt \quad (175)$$

The logarithmic exponential integral

$$e^{-\eta} \int_{-\infty}^{\eta} \frac{e^{t}}{t} \log\left(1 + \frac{t}{\epsilon}\right) dt \quad (176)$$

and the arctangential Fresnel integral

$$e^{-\eta} \int_{-\infty}^{\eta} \frac{e^{t}}{t^{\frac{1}{2}}} \tan^{-1}\left(\frac{t}{\epsilon}\right)^{\frac{1}{2}} dt \quad (177)$$

are obtained from subroutines[34]. It is necessary to apply corrections to bring the path of integration below each singularity at $t = -\epsilon$. For the logarithmic exponential integral the correction is

$$- 2\pi i\, e^{-\eta} \int_{-\infty}^{-\epsilon} \frac{e^t}{t}\, dt + 2\pi i\, e^{-\eta} \int_{-\infty}^{\eta} \frac{e^t}{t}\, dt \qquad (178)$$

and for the arctangential Fresnel integral the correction is

$$- \pi\, e^{-\eta} \int_{-\infty}^{-\epsilon} \frac{e^t}{t^{\frac{1}{2}}}\, dt + \pi\, e^{-\eta} \int_{-\infty}^{\eta} \frac{e^t}{t^{\frac{1}{2}}}\, dt \qquad (179)$$

In the evaluation of the ascending series the arctangential Fresnel integral is corrected only when

$$\mathcal{R}e\, \epsilon \geq 0 \quad \text{and} \quad \mathcal{I}m\, \epsilon > 0 \quad \text{or} \quad \mathcal{R}e\, \epsilon < 0 \quad \text{and} \quad \mathcal{I}m\left(i + \frac{\epsilon}{\eta}\right) \geq 0 \qquad (180)$$

In the evaluation of the descending series the integrals are corrected only when

$$\mathcal{R}e\, \epsilon > 0 \quad \text{and} \quad \mathcal{I}m\, \epsilon > 0 \qquad (181)$$

When the differences are taken between values for the upper and lower limits of ϵ the correction for the logarithmic exponential integral is reduced to

$$- 2\pi i\, e^{-\eta} \int_{-\infty}^{-\epsilon} \frac{e^t}{t}\, dt \qquad (182)$$

and the correction for the arctangential Fresnel integral is reduced to

$$- \pi\, e^{-\eta} \int_{-\infty}^{-\epsilon} \frac{e^t}{t^{\frac{1}{2}}}\, dt \qquad (183)$$

The arctangential Fresnel integral and the arccotangential Fresnel integral are related in accordance with the equation

$$e^{-\eta} \int_{-\infty}^{\eta} \frac{e^t}{t^{\frac{1}{2}}} \tan^{-1}\left(\frac{\epsilon}{t}\right)^{\frac{1}{2}} dt = \frac{\pi}{2}\, e^{-\eta} \int_{-\infty}^{\eta} \frac{e^t}{t^{\frac{1}{2}}}\, dt - e^{-\eta} \int_{-\infty}^{\eta} \frac{e^t}{t^{\frac{1}{2}}} \tan^{-1}\left(\frac{t}{\epsilon}\right)^{\frac{1}{2}} dt \qquad (184)$$

which is used in connection with the evaluation of the ascending series.

Accuracy and Efficiency

The integration by parts gives acceptable accuracy and efficiency everywhere. Errors and times are illustrated by solid curves in Figures 5 and 6.

PROGRAMMING

Trapezoidal integration for velocity potential is provided by Subroutines CKPSVP and CEXPLI, while asymptotic approximation and integration by parts are provided by

SUBROUTINE PSWTVP (AK, AH, AX, AY, AZ, FP)

****

FORTRAN SUBROUTINE FOR VELOCITY POTENTIAL OF POINT SOURCE

****

The critical wave number κ_0 is given in argument AK, and the depth h of the source is given in argument AH. The coordinates x, y, z of a point are given in arguments AX, AY, AZ. The velocity potential φ is placed in function FP. Calls are made to auxiliary subroutines which are listed in Table I.

Trapezoidal integration for velocity field is provided by Subroutines CKPSVF and CEXPDI, while asymptotic approximation and integration by parts are provided by

SUBROUTINE PSWTVF (AK, AH, AX, AY, AZ, FU, FV, FW)

****

FORTRAN SUBROUTINE FOR VELOCITY FIELD OF POINT SOURCE

****

The critical wave number κ_0 is given in argument AK, and the depth h of the source is given in argument AH. The coordinates x, y, z of a point are given in arguments AX, AY, AZ. The components u, v, w of velocity are placed in functions FU, FV, FW. Calls are made to auxiliary subroutines which are listed in Table I.

TABLE I

AUXILIARY SUBROUTINES

Name	Function
CVCTRM	Performs complex vector-matrix multiplication.
CLGRNP	Synthesizes complex Lagrangian polynomials.
CBSSLK	Computes the modified Bessel function of the second kind.
CLGMCI	Computes the complex logarithmic integral.
CEXPLI	Computes the complex exponential integral.
CFRNLI	Computes the complex Fresnel integral.
CEXEXI	Computes the complex exponential exponential integral.
CEXFRI	Computes the complex exponential Fresnel integral.
CLGEXI	Computes the logarithmic exponential integral.
CATFRI	Computes the arctangential Fresnel integral.

DISCUSSION

The fundamental formulation of the Green function can be transformed into other formulations in which the added Fourier integrals have various amplitudes. Methods of evaluation by contour integration in the complex plane of a coordinate in wave number space have been published in the literature. The final evaluation in each method is an application of a quadrature rule to the azimuthal integration. Any systematic quadrature rule other than the trapezoidal rule for equally spaced arguments is equivalent to a combination of trapezoidal rules of reduced order. It can have superior accuracy only by accident.

Radial integration along a complex path was suggested originally by Pond[7] in 1957, and was programmed for computation by DiDonato[7] in 1958. In the DiDonato method, the amplitude had its original form as expressed by the equation

$$A(\kappa, \theta) = \frac{1}{2\pi\kappa} \frac{(\kappa_0 + \kappa \cos^2\theta)}{(\kappa_0 - \kappa \cos^2\theta)} \tag{185}$$

Radial integration was in the complex plane of the radial coordinate. Contour integration and evaluation of the residue of a pole led to the sum of a single integral and a double integral. The single integral had a variable upper limit. The radial integration was by Simpson rule, and the azimuthal integration was by Gauss rule.

Radial integration along a complex path was suggested again by Smith, Giesing, and Hess[8] in 1963, and by Webster[9] in 1969. It was programmed for computation by Adee[9] in 1973. In the Adee method, the amplitude is resolved as expressed by the equation

$$A(\kappa, \theta) = -\frac{1}{2\pi\kappa} + \frac{\kappa_0}{\pi\kappa(\kappa_0 - \kappa \cos^2\theta)} \tag{186}$$

The first term of the expansion is the amplitude of a negative image source. Radial integration in wave number space was performed along a complex path where the integrand is not oscillatory. Radial integration was by series expansion and Simpson rule, then azimuthal integration was completed by Simpson rule. The radial integration evaluated just the conventional exponential integral, and the azimuthal integration was applied to the same integrand as in the case of the trapezoidal rule.

A formulation of the Green function has been given by Gadd[10] in 1970. In the Gadd method, the amplitude is resolved in accordance with the alternative equation

$$A(\kappa, \theta) = +\frac{1}{2\pi\kappa} + \frac{\cos^2\theta}{\pi(\kappa_0 - \kappa \cos^2\theta)} \tag{187}$$

The first term of the expansion is the amplitude of a positive image source. As $\kappa \to \infty$ in this method the integrand does not diminish as fast as in the conventional method. If the source is close to the surface, a large part of the integration is devoted to the computation of twice the potential of the negative image. Slow convergence of the integrand makes accurate evaluation difficult.

That radial integration evaluates an exponential integral was recognized[21,22] in 1959. The exponential integral was used in subroutines which were reported[16] in 1965. That direct evaluation of the exponential integral is more efficient than the conventional radial integration was appreciated by Shen and Farell[11] in 1975. In the Shen and Farell method, the exponential integral is evaluated by reference to a subroutine. In the azimuthal integration, the angle θ is replaced by its tangent t. The azimuthal integration is completed by Simpson rule. The computer is required to halve the interval of integration until the integral remains constant to within a tolerance. Except for a factor $d\theta/dt$, the integrand is the same as the integrand for the trapezoidal integration.

The integration in wave number space has been analysed in terms of Cartesian coordinates by Andersson[12,13] and by Noblesse[14] in 1976. In the Noblesse method, contour integration in the complex plane of the longitudinal Cartesian coordinate leads to a resolution of the Green function into a monotonic term, which is symmetric fore and aft, and an oscillatory term, which trails behind the source. The monotonic term has an exponential integral in its integrand, but the argument of the exponential integral is not the same as the conventional argument. The real part of the argument changes sign but the imaginary part remains negative during azimuthal integration. The argument of the exponential integral goes to zero at the limits of integration, and the exponential integral presents there a pair of logarithmic singularities. Subtraction of a logarithm from the integrand gives a function more suitable for quadrature, then the logarithm is restored with analytic integration. The oscillatory term is just twice the conventional single integral, and presents the same problem of evaluation. Although the sum of the monotonic term and the oscillatory term is continuous, the individual terms have discontinuities in derivative at the transverse plane through the source. They do not satisfy the free-surface boundary conditions at this transverse plane.

An attempt to compare the present method with the various other methods has been thwarted by an inability to make the programming of the other methods run properly on the computer in this laboratory.

An effort has been made to interpolate[18] in a table of velocities. A table of 36900 entries was computed on the Naval Ordnance Research Calculator when that computer was waiting to be demolished. The interpolation was founded on the assumption that the velocities could be expressed as the product of a monotonic function and the exponential function of a monotonic argument. This is true where the asymptotic approximation is valid, but close to the source the divergent wave system and the transverse wave system are inextricably interlocked. Efforts to interpolate had only indifferent success. In order to interpolate the table would have to be so large as to be unwieldy. The three-way interpolation could be as costly as the direct evaluation.

Measurements of wave height of a Rankine ovoid[25-27] confirm the validity of computations of wave height by subroutine. The Rankine ovoid is that streamline which is generated by a simple source and sink on a line parallel to the free stream. A model of the Rankine ovoid has been towed in the model basin and wave heights have been measured by Shaffer[25]. The measured wave heights agreed in phase with the computed wave heights but were smaller in amplitude. The model was towed by a long stranded cable, which undoubtedly set up a fully turbulent boundary layer on the model. Inasmuch as vorticity in the boundary layer would meet partially the boundary conditions on the boundary of the ovoid, it was necessary for the source distribution to meet only partially the boundary conditions. The agreement between measured and computed wave heights is excellent in view of the experimental difficulties.

CONCLUSION

The trapezoidal integration is efficient in the near field of a deep source, but is inefficient in the far field of a shallow source. The asymptotic approximation is efficient far from the source, but is inaccurate along the critical line of wave crests. The integration by parts is most uniformly accurate and efficient at all depths.

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APPENDIX A

TRANSIENT WAVE

TRANSIENT WAVE

In the analysis of a transient wave, the uniform motion of a source along a horizontal line is simulated by a succession of pulses at equal intervals of distance and time.

Let x, y, z, t be coordinates and time with origin of coordinates at the surface of the fluid initially at rest. Let a fixed source be created at a distance h below the origin. The velocity is the negative gradient $-\nabla\varphi$ of a velocity potential φ, which is a solution of Laplace's equation. The dynamical equation is the complete Bernoulli equation,

$$-\frac{\partial\varphi}{\partial t} + \tfrac{1}{2}(\nabla\varphi)^2 + \frac{p}{\rho} \quad gz = \text{constant} \tag{1}$$

where ρ is the density, p is the pressure, and g is the acceleration of gravity. Let the surface of the fluid be given by the equation

$$z + \zeta(x, y, t) = 0 \tag{2}$$

where ζ is the elevation of the surface. The linearized Bernoulli equation is

$$-\frac{\partial\varphi}{\partial t} + g\zeta = 0 \tag{3}$$

at the free surface, and the kinematic equation is

$$-\frac{\partial\varphi}{\partial z} + \frac{\partial\zeta}{\partial t} = 0 \tag{4}$$

Elimination of ζ from Equations (3) and (4) leads to the boundary equation

$$\frac{\partial^2\varphi}{\partial t^2} - g\,\frac{\partial\varphi}{\partial z} = 0 \tag{5}$$

which must be satisfied by the transient potential.

At the instant of creation of the source the fluid is accelerated into motion. All terms in the Bernoulli equation remain bounded at the free surface. If each term is integrated with respect to time, then the term $\partial\varphi/\partial t$ becomes the change of potential, but the other terms vanish. In the limit of instantaneous formation of the source, the potential remains constant, and the boundary conditions require that the formation of a source beneath the surface be accompanied by the formation of an image over the surface.

The potential of the surface disturbance is given by the equation

$$\varphi(x, y, z, t) = \varphi_1(x, y, z) + \varphi_2(x, y, z) + \varphi_3(x, y, z, t) \tag{6}$$

where φ_1 is the potential of the source in an unbounded fluid, φ_2 is the potential of the image source over the free surface, and φ_3 is the potential of a transient disturbance.

For a unit source below the surface the potential φ_1 is given by the equation

$$\varphi_1 = + \frac{1}{\sqrt{x^2 + y^2 + (z - h)^2}} \tag{7}$$

and for the image source over the surface the potential φ_2 is given by the equation

$$\varphi_2 = - \frac{1}{\sqrt{x^2 + y^2 + (z + h)^2}} \tag{8}$$

The Fourier transforms of the three potentials are given by the equations

$$\varphi_1 = + \frac{1}{2\pi} \int_{-\pi}^{+\pi} \int_0^{\infty} e^{-\kappa|z-h| + i\kappa(x\cos\theta + y\sin\theta)} \, d\kappa \, d\theta \tag{9}$$

$$\varphi_2 = - \frac{1}{2\pi} \int_{-\pi}^{+\pi} \int_0^{\infty} e^{-\kappa(z+h) + i\kappa(x\cos\theta + y\sin\theta)} \, d\kappa \, d\theta \tag{10}$$

$$\varphi_3 = \int_{-\pi}^{+\pi} \int_0^{\infty} A(\kappa, \theta, t) \, e^{-\kappa(z+h) + i\kappa(x\cos\theta + y\sin\theta)} \, \kappa \, d\kappa \, d\theta \tag{11}$$

Substitution of potentials in the free-boundary equation shows that the Fourier amplitude satisfies the equation

$$\frac{\partial^2 A}{\partial t^2} + g\kappa A - \frac{g}{\pi} = 0 \tag{12}$$

of which the appropriate solution is

$$A(\kappa, \theta, t) = \frac{1}{\pi\kappa} - \frac{\cos\sqrt{g\kappa}\, t}{\pi\kappa} \tag{13}$$

If an instantaneous creation of a finite source is followed after a time interval dt by the instantaneous annihilation of the finite source, then the Fourier amplitude per unit pulse is merely

$$A(\kappa, \theta, t) = \frac{1}{\pi} \sqrt{\frac{g}{\kappa}} \sin\sqrt{g\kappa}\, t \tag{14}$$

The potentials φ_1 and φ_2 are zero for the unit pulse, while the potential φ_3 is given by the equation

$$\varphi_3 = \frac{1}{\pi} \int_{-\pi}^{+\pi} \int_0^{\infty} \sqrt{g\kappa} \sin\sqrt{g\kappa}\, t \, e^{-\kappa(z+h) + i\kappa(x\cos\theta + y\sin\theta)} \, d\kappa \, d\theta \tag{15}$$

Substitution of the potential in the boundary equation shows that the surface elevation initially is a hump, but breaks ultimately into the well known system of concentric waves.

If a unit source is created below the origin at depth h and then moves along the x-axis at constant speed U, the potentials φ_1 and φ_2 are restored but with their origins centered over the source. The potential φ_3 may be obtained through integration of the contributions of pulses which have been created during a succession of differential time intervals $d\tau$ from $\tau = 0$ to $\tau = t$. The result at time t is given by the equation

$$\varphi_3 = \frac{1}{\pi} \iiint \sqrt{g\kappa} \sin \sqrt{g\kappa}\,(t - \tau)\, e^{-\kappa(z+h)+i\kappa[(x-U\tau)\cos\theta + y\sin\theta]}\, d\kappa\, d\theta\, d\tau \tag{16}$$

After integration the potential is given by the equation

$$\begin{aligned}
\varphi_3 = &- \frac{i}{2\pi} \iint \frac{\sqrt{g\kappa} \sin(\sqrt{g\kappa} + \kappa U \cos\theta)t}{\sqrt{g\kappa} + \kappa U \cos\theta}\, e^{-\kappa(z+h)+i\kappa[(x-Ut)\cos\theta + y\sin\theta]}\, d\kappa\, d\theta \\
&+ \frac{i}{2\pi} \iint \frac{\sqrt{g\kappa} \sin(\sqrt{g\kappa} - \kappa U \cos\theta)t}{\sqrt{g\kappa} - \kappa U \cos\theta}\, e^{-\kappa(z+h)+i\kappa[(x-Ut)\cos\theta + y\sin\theta]}\, d\kappa\, d\theta \\
&+ \frac{1}{\pi} \iint \frac{\sqrt{g\kappa} \sin^2\frac{1}{2}(\sqrt{g\kappa} + \kappa U \cos\theta)t}{\sqrt{g\kappa} + \kappa U \cos\theta}\, e^{-\kappa(z+h)+i\kappa[(x-Ut)\cos\theta + y\sin\theta]}\, d\kappa\, d\theta \\
&+ \frac{1}{\pi} \iint \frac{\sqrt{g\kappa} \sin^2\frac{1}{2}(\sqrt{g\kappa} - \kappa U \cos\theta)t}{\sqrt{g\kappa} - \kappa U \cos\theta}\, e^{-\kappa(z+h)+i\kappa[(x-Ut)\cos\theta + y\sin\theta]}\, d\kappa\, d\theta
\end{aligned} \tag{17}$$

In the limit as $t \to \infty$, the first integral vanishes unless $\cos\theta$ is negative, and the second integral vanishes unless $\cos\theta$ is positive. In either case there is no contribution to the integration except where the denominators vanish. The substitution

$$\kappa = \frac{g}{U^2\cos^2\theta} + u \tag{18}$$

and use of the equation

$$\int_0^\infty \frac{\sin au}{u}\, du = \frac{\pi}{2} \qquad (a > 0) \tag{19}$$

for which a is given by the equation

$$a = \tfrac{1}{2}Ut|\cos\theta| \tag{20}$$

converts the sum of the first two integrals into the single integral in Equation (34) of the text. In the second two integrals the integrands are multiplied by the square of a sine which is always positive and in the limit as $t \to \infty$ has the effect of weighting the Cauchy principal value by $\frac{1}{2}$. The sum of the second two integrals becomes the double integral in Equation (34) of the text. Equation (17) is very instructive insofar as it demonstrates the true nature of the Cauchy principal value.

APPENDIX B

ALGORITHMS

ALGORITHMS

Integration of Series

Let a function $A(t)$ be expressed by the equation

$$A(t) = \sum_{k=0}^{N} a_k t^k \tag{1}$$

The nth integral $A^{(n)}(t)$ is given by the definition

$$A^{(n)}(t) = \sum_{k=0}^{N} a_k^{(n)} t^k \tag{2}$$

for which the coefficients are derived from the recurrence

$$a_k^{(n)} = \frac{1}{k} a_{k-1}^{(n-1)} \tag{3}$$

The coefficients are stored in an array and are shifted during each cycle of recurrence. The constant of integration $a_0^{(n)}$ is replaced by a new constant in each cycle.

Differentiation of Quotient

Let a function $Q(t)$ be expressed by the equation

$$Q(t) = \frac{\sum a_k t^k}{\sqrt{1 + t^2}} \tag{4}$$

The nth derivative $Q^{(n)}(t)$ is given by the definition

$$Q^{(n)}(t) = \frac{\sum a_k^{(n)} t^k}{(1 + t^2)^{n+\frac{1}{2}}} \tag{5}$$

for which the coefficients are derived from the recurrence

$$a_k^{(n)} = (k - 2n) a_{k-1}^{(n-1)} + (k + 1) a_{k+1}^{(n-1)} \tag{6}$$

The coefficients are stored in an array and are replaced during each cycle of recurrence.

Transformation of Variable

If a dependent variable is expressed in terms of either of two independent variables by power polynomials, then the equality of the power polynomials defines implicitly one independent variable in terms of the other, provided the power polynomials meet necessary restrictions. Let P be the dependent variable and let t, u be the independent variables. Let the variable P be given either by the equation

$$P = \sum_{m=1}^{M} a_m t^m \tag{7}$$

or by the equation

$$P = \sum_{n=1}^{N} b_n u^n \tag{8}$$

Equivalence of the two polynomials implies that t and u are related by the equation

$$t = \sum_{k=1}^{\infty} c_k u^k \tag{9}$$

All of these polynomials are restricted to have zero constant terms.

The mth power of t is given by the definition

$$t^m = \sum_{n=m}^{\infty} c_n^{(m)} u^n \tag{10}$$

The coefficients $c_n^{(m)}$ all are zero for $n < m$ and the first nonzero coefficient is given by the equation

$$c_m^{(m)} = (c_1)^m \tag{11}$$

The coefficients are generated by the recurrence

$$c_n^{(m)} = \sum_{k=1}^{n-m+1} c_k^{(1)} c_{n-k}^{(m-1)} \tag{12}$$

The expression for $c_n^{(m)}$ contains no coefficient $c_k^{(1)}$ of higher order than $c_{n-m+1}^{(1)}$.

The coefficients are solutions of the equations

$$\sum_{m=1}^{n} a_m c_n^{(m)} = b_n \tag{13}$$

which express the equivalence of the polynomials for P. Solution of the equations is achieved by an iteration which starts with all coefficients c_n set equal to zero except c_1. When a_1 and b_1 both are zero, the coefficient c_1 is given by the equation

$$c_1 = (b_2/a_2)^{\frac{1}{2}} \tag{14}$$

The coefficients $c_{n-1}^{(1)}$ of higher order are obtained directly from the equation

$$2a_2 c_1 c_{n-1}^{(1)} = b_n - \sum_{m=2}^{n} a_m c_n^{(m)} \tag{15}$$

whence the coefficient $c_n^{(2)}$ is adjusted in accordance with the transformation

$$c_n^{(2)} \rightarrow c_n^{(2)} + 2c_1 c_{n-1}^{(1)} \tag{16}$$

When a_1 and b_1 are not zero, the coefficients $c_n^{(1)}$ of progressively increasing order are obtained directly from the equation

$$a_1 c_n^{(1)} = b_n - \sum_{m=2}^{n} a_m c_n^{(m)} \tag{17}$$

Thus the coefficients are generated by a straightforward progression.

The integration of a Taylor series in t is converted into an integration of a power series in u with the aid of the equations

$$t^m \frac{dt}{du} = \frac{1}{m+1}\frac{d}{du}t^{m+1} = \frac{1}{m+1}\sum_{n=m}^{\infty}(n+1)c_{n+1}^{(m+1)}u^n \tag{18}$$

No further multiplication by power series is required for the change of differential in the integration. The coefficients of the polynomials which express the powers of t in terms of the powers of u are stored in a pair of matrices.

Airy Function

The Airy function $Ai(z)$ is defined by the equation

$$Ai(z) = \frac{1}{\pi}\int_0^{\infty}\cos(zt + \tfrac{1}{3}t^3)\,dt \tag{19}$$

or by the equation

$$Ai(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(zt+\frac{1}{3}t^3)}\,dt \tag{20}$$

The first derivative $Ai'(z)$ is given by the equation

$$Ai'(z) = \frac{i}{2\pi}\int_{-\infty}^{+\infty} t e^{i(zt+\frac{1}{3}t^3)}\,dt \tag{21}$$

and the second derivative $Ai''(z)$ is given by the equation

$$Ai''(z) = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} t^2 e^{i(zt+\frac{1}{3}t^3)}\,dt \tag{22}$$

Addition of z to t^2 in the integrand changes it into a function which can be integrated in finite terms as expressed by the equation

$$Ai''(z) = \frac{z}{2\pi}\int_{-\infty}^{+\infty} e^{i(zt+\frac{1}{3}t^3)}\,dt + \frac{i}{2\pi}e^{i(zt+\frac{1}{3}t^3)}\Big|_{-\infty}^{+\infty} \tag{23}$$

The integration converges if the limits of integration lie above the real axis, in which case the Airy function is a solution of the differential equation

$$Ai''(z) = zAi(z) \tag{24}$$

Let the nth derivative $Ai^{(n)}(z)$ be expressed by the equation

$$Ai^{(n)}(z) = \sum a_k^{(n)} z^k Ai(z) + \sum b_k^{(n)} z^k Ai'(z) \tag{25}$$

The coefficients of progressively higher order are generated by the recurrence equations

$$a_k^{(n)} = (k+1) a_{k+1}^{(n-1)} + b_{k-1}^{(n-1)} \tag{26}$$

$$b_k^{(n)} = a_k^{(n-1)} + (k+1) b_{k+1}^{(n-1)} \tag{27}$$

which are started with the single coefficient of zero order, $a_0^{(0)} = 1$.

The expression of the Airy function in terms of Bessel functions is derived on page 188 of Watson's *Theory of Bessel Functions*[20].

The exponential function in the integrand of the Airy function can be expanded in an absolutely convergent power series in z. If the path of integration with respect to t is varied to pass inward toward the origin along a ray at the phase angle $\frac{5}{6}\pi$, and outward from the origin along a ray at the phase angle $\frac{1}{6}\pi$, then the coefficients of the power series can be expressed in terms of gamma functions and can be correlated with the coefficients in the expansions of Bessel functions of imaginary argument.

The Airy function is given in terms of the McDonald function by the equation

$$Ai(z) = + \frac{z^{\frac{1}{2}}}{\pi\sqrt{3}} K_{\frac{1}{3}}(\tfrac{2}{3} z^{\frac{3}{2}}) \tag{28}$$

and its derivative is given by the equation

$$Ai'(z) = - \frac{z}{\pi\sqrt{3}} K_{\frac{2}{3}}(\tfrac{2}{3} z^{\frac{3}{2}}) \tag{29}$$

Evaluation of the McDonald function is performed by reference to a subroutine[23,24] for the complex Bessel function.

The argument of z is in the range 0 to 2π, whereas the argument for the subroutine is in the range $-\pi$ to $+\pi$. If the argument exceeds the range of the subroutine, then the McDonald function is modified in such a way as to increase the argument by 2π.

APPENDIX C

FIGURES

WAVE PROFILE

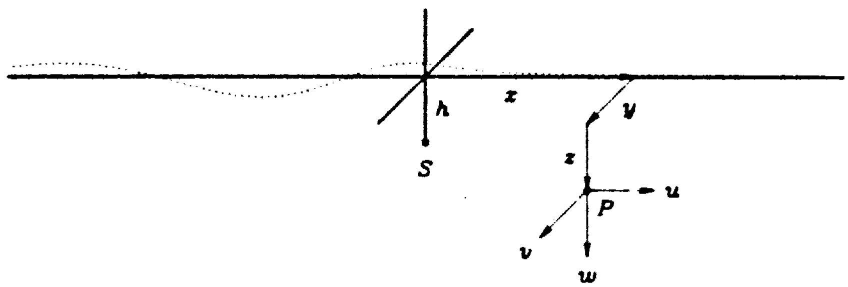


Figure 1. Cartesian Coordinates. Position of a point P with respect to the source S.

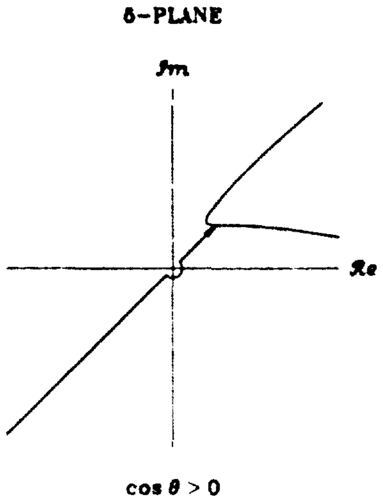


Figure 2. Radial Integration with respect to Wave Number. —, path of integration; —, limit of integration.

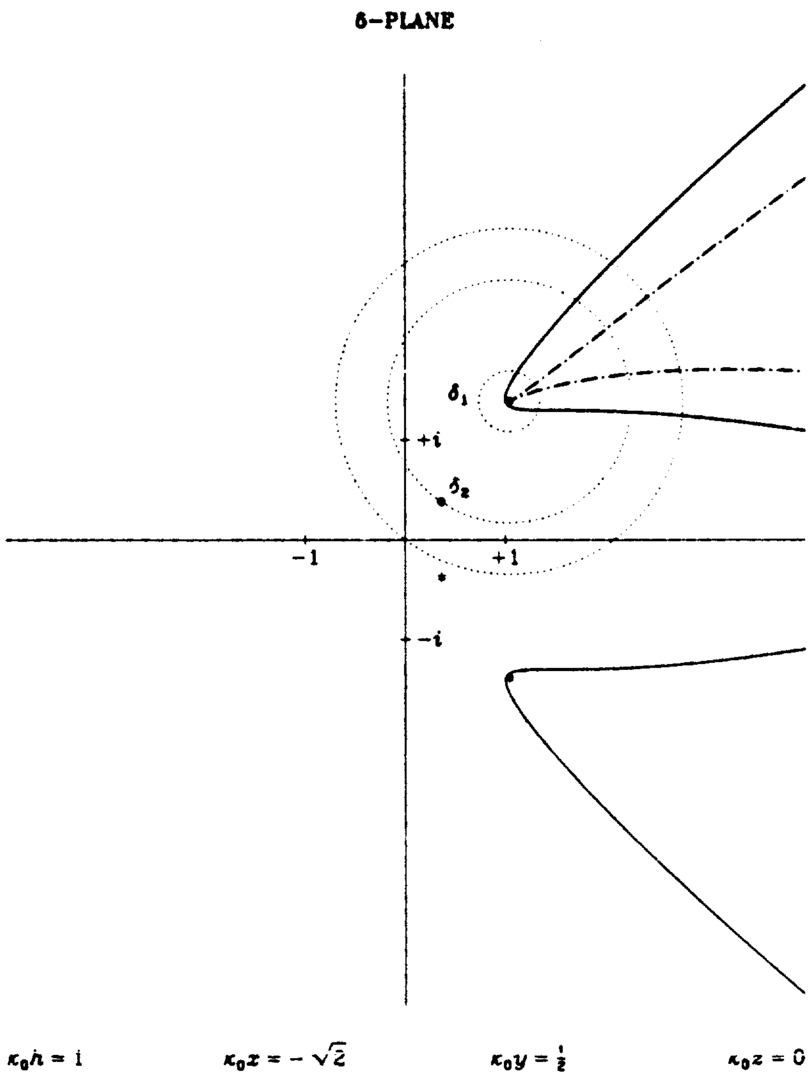
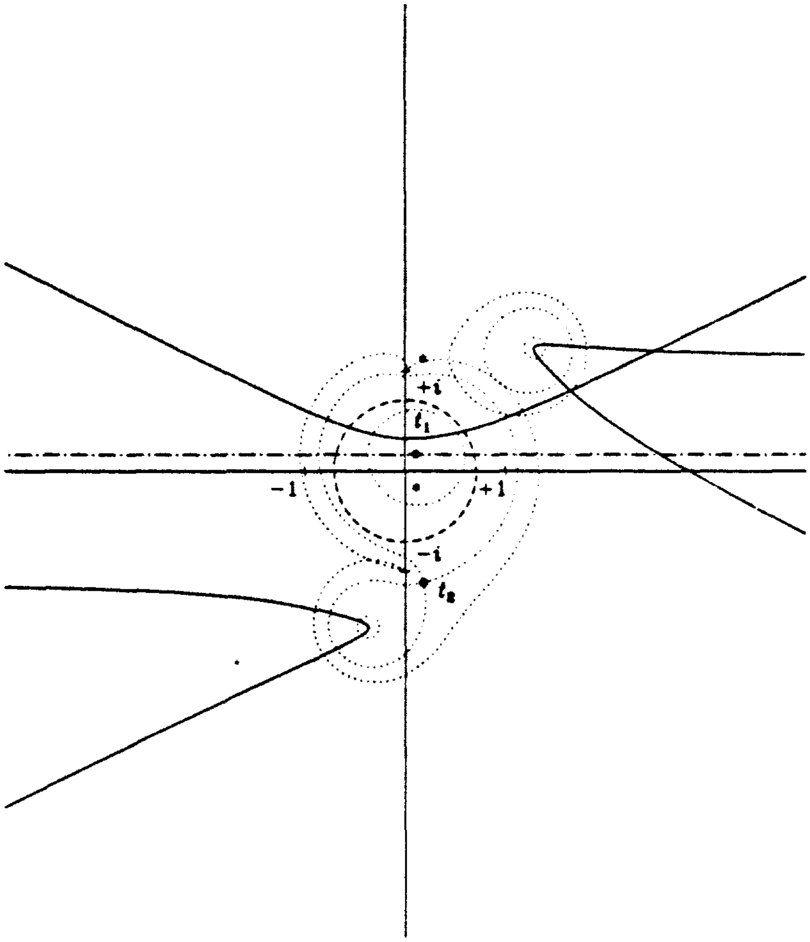


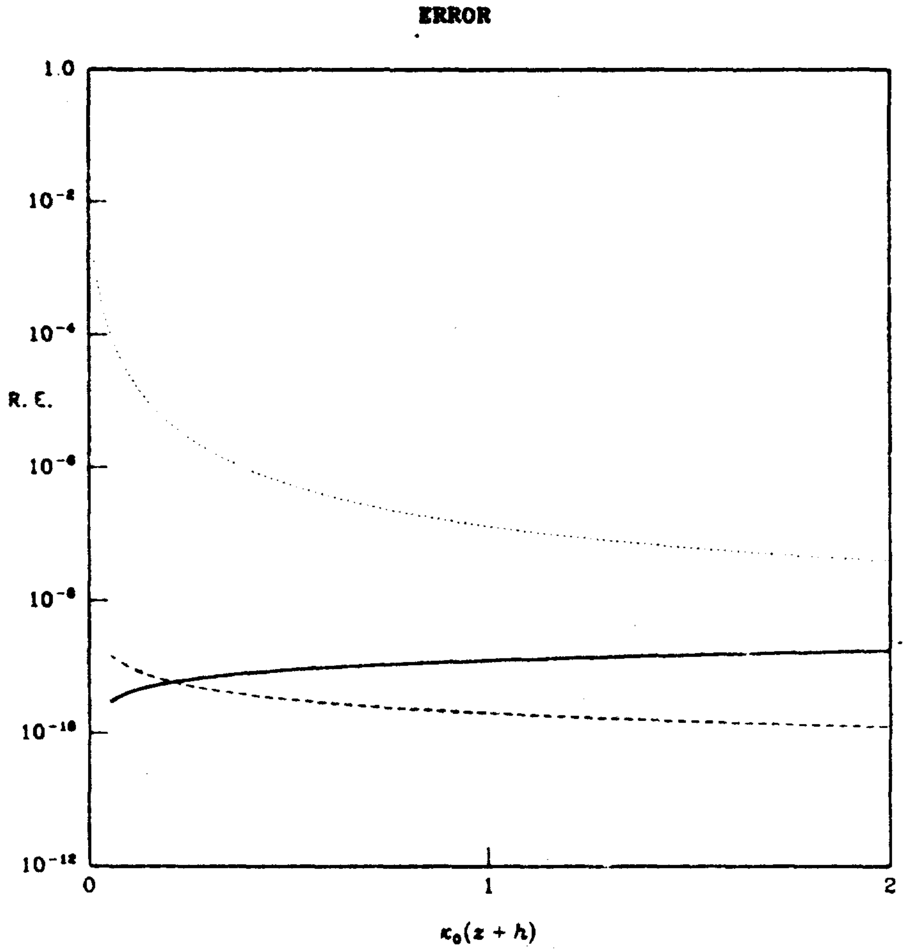
Figure 3. Azimuthal Integration with respect to Wave Number. •, primary singularities; •, secondary singularities; ···, circle of convergence; —, primary branch, natural path; —, secondary branch, natural path; —·—, actual path of integration.

t-PLANE



$\kappa_0 h = 1$ $\kappa_0 x = -\sqrt{2}$ $\kappa_0 y = \frac{1}{2}$ $\kappa_0 z = 0$

Figure 4. Azimuthal Integration with respect to Wave Number. •, primary singularities; •, secondary singularities; – – –, circle of radius $\frac{1}{\sqrt{2}}$; ···, boundary of convergence; —, primary branch, natural path; —, secondary branch, natural path; –·–, actual path of integration.



$x = -\sqrt{8}y$ $\qquad$ $y = \frac{1}{3}\sqrt{x^2 + y^2}$

Figure 5. Relative Error along the Critical Line of Wave Crests. ---, trapezoidal integration; ···, asymptotic approximation; —, integration by parts.

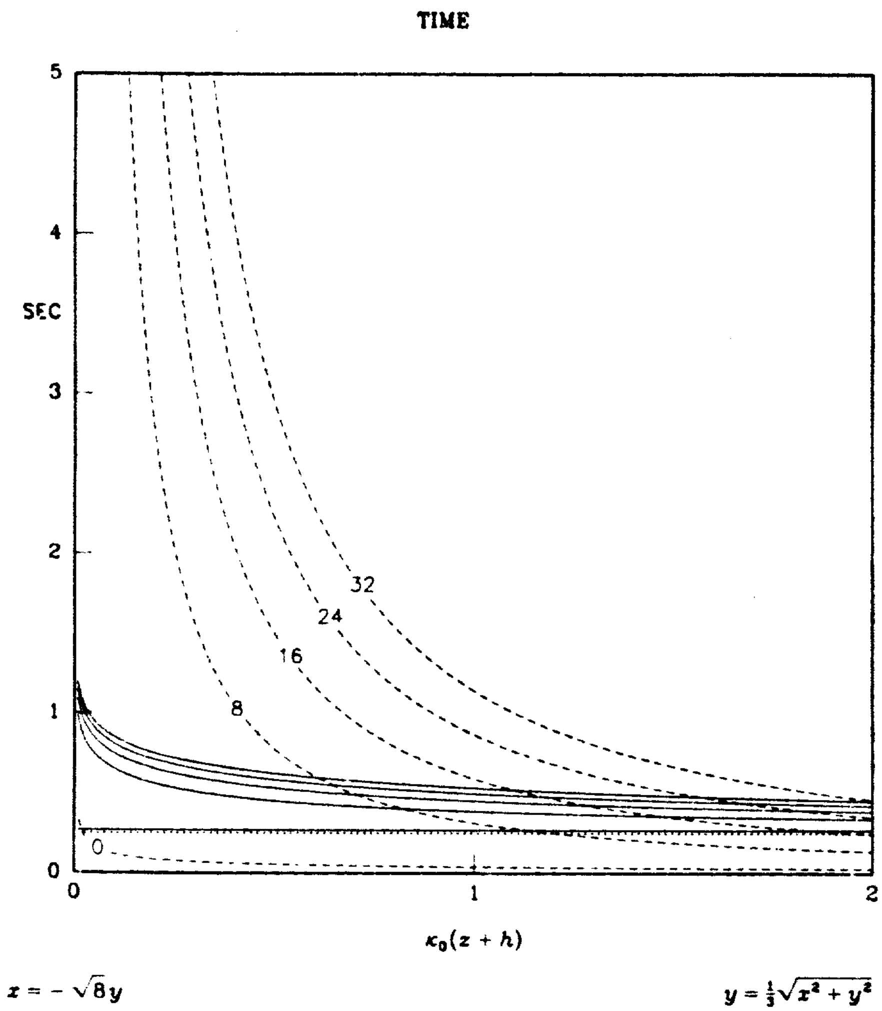


Figure 6 Computation Time along the Critical Line of Wave Crests. Curves for $\kappa_0\sqrt{x^2+y^2}$ equal to 0(8)32. ---, trapezoidal integration; ···, asymptotic approximation; —, integration by parts.

APPENDIX D

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4. TITLE (and Subtitle) COMPUTATION OF VELOCITY IN THE WAVE TRAIN OF A POINT SOURCE		5. TYPE OF REPORT & PERIOD COVERED Final
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) A. V. Hershey		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Surface Weapons Center Dahlgren Laboratory (DK-05) Dahlgren, Virginia 22448		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NIF
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Surface Weapons Center Dahlgren Laboratory Dahlgren, Virginia 22448		12. REPORT DATE September 1977
		13. NUMBER OF PAGES 58
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

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18. SUPPLEMENTARY NOTES

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fluid flow velocities
velocity components, flow

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The velocity potential and the components of velocity in the wave train of a point source are derived from Havelock's integral. Analysis and documentation are given for the computation of velocity by trapezoidal integration, by asymptotic approximation, and by integration by parts. Accuracy and efficiency are least dependent upon the depth of the source in the integration by parts. The three methods of computation are available for applications in a set of subroutines.

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